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Semistable sheaves on homogeneous spaces and abelian varieties

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Abstract. In this paper we prove that semistable sheaves with zero Chern classes or homogeneous spaces are trivial and semistable sheaves on abelian varieties with zero Chern classes are filtered by line bundles numerically equivalent to zero. The method consists in reducing mod p and then showing that the Frobenius morphism preserves semistability on the above class of varieties. For technical reasons, we have to assume boundedness of semistable sheaves in char p.

Keywords. Semistable sheaf; homogeneous space; abelian variety; Frobenius morphism purely inseparable descent.

Let X be a smooth projective variety of dimension d defined over an algebraically closed field k and equipped with a very ample line bundle H. For a coherent sheaf F on X we denote by $F(n)$ the sheaf $F \otimes H^n$ and by $\chi_F(n)$ the Hilbert polynomial of F, $\chi_F(n) = \chi[F(n)]$. Let $c_1(F)$ denote the first Chern class of F. If F is torsion-free, then a subsheaf E of F is defined to be a *proper* subsheaf of F if $0 < rkE < rkF$. Define

$$\mu(F) = \frac{c_1(F) \cdot H^{d-1}}{rkF}$$

and

$$p_F(n) = \frac{\chi_F(n)}{rkF}$$

A torsion-free sheaf V on X is defined to be *μ-stable* (resp. *μ-semistable*) if for all subsheaves W of V we have (Takemoto [22])

$$\mu(W) < \mu(V) \text{ (resp. } \mu(W) \leqslant \mu(V))$$

V is defined to be χ-stable (resp. χ-semistable) if for all proper subsheaves W of V we have

$$p_W(n) < p_V(n) \text{ (resp. } p_W(n) \leqslant p_V(n))$$

for all $n \gg 0$ (Gieseker [6]; Maruyama [13]).

Fix a *Weil* cohomology $X \to H^*(X)$ (cf. [10]) and let $\mathscr{S} = \{V, r, c_1 \ldots, c_d\}$ denote the set of all χ-semistable torsion-free sheaves on X of rank r and fixed Chern classes $c_i = c_i(V)$, $1 \leqslant i \leqslant d$, where $c_i(V) \in H^*(X)$. A basic problem which arises in the construction of the moduli spaces of these sheaves, is the *boundedness* of $\mathscr{S}$ (cf. [13]). If char $k = 0$ then the boundedness of $\mathscr{S}$ has been proved in [5]. If char $k = p > 0$, then

the boundedness of $\mathscr{S}$ has been proved only in some special cases: dim $X = 2$, $rk\, V$ arbitrary in [14], and dim X arbitrary and $rk\, V < 3$ in [15].

In this paper we make the following assumption:

*{The set $\mathscr{S}$ is bounded for all values of dim X and $rk\, V$.}

Using * we prove

Theorem 1: Let X be a complete homogeneous space under a reductive algebraic group and let V be a χ-semistable torsion free sheaf on X. Then if $c_i(V) = 0$ for $1 \leqslant i \leqslant d$ we have V is *trivial i.e.* $V \simeq \mathbf{O}_X^r$, where $r = rk\, V$.

Theorem 2: Let X be an abelian variety and V a χ-semistable torsion-free sheaf on X. If $c_i(V) = 0$ for $1 \leqslant i \leqslant d$, then there is a filtration of V

$$0 = V_0 \subset V_1 \ldots \subset V_r = V$$

where each V_i/V_{i-1}, $1 \leqslant i \leqslant r$, is a *line bundle* on X *algebraically equivalent* to zero. In particular V is *locally free.*

Combining theorems 1 and 2 we get

Theorem 3: Let G be a smooth algebraic group acting transitively on a smooth projective variety X such that for any $x \in X$, the canonical map $G \to X$ given by $g \to gx$ is smooth. Let V be a χ-semistable torsion-free sheaf on X with $c_i(V) = 0$ for $1 \leqslant i \leqslant \dim X$. Then the conclusion of theorem 2 holds for V, *i.e.* V has a filtration by subsheaves whose successive quotients are line bundles algebraically equivalent to zero.

Theorem 1 for rank 2 semistable bundles on $\mathbf{P}^2$ or on $\mathbf{P}^1 \times \mathbf{P}^1$ has been proved by several authors (Takemoto [22]; Maruyama [16]; Schwarzenberger [28]). These results depend on the Riemann-Roch theorem for surfaces. For rank 2 semistable bundles on $\mathbf{P}^n$ in characteristic zero theorem 1 can also be deduced from the results of Ellencwajg-Forster [4] and Maruyama [14].

Theorem 2 for rank 2 bundles on abelian surface has been proved by Umemura [29] and Takemoto [22], and for bundles of arbitrary rank on abelian surfaces by Mukai [25].

Theorem 2 can be generalized as follows: (a) A vector bundle V on an abelian variety is called *weakly-translation invariant* (*semi-homogeneous* in the sense of Mukai [24]) if $T_x V \simeq V \otimes L_x$ for all $x \in X$, where L_x is a line bundle depending on x.

(b) A vector bundle V on X is said to be *special* if there exists an isogeny: $f\colon Y \to X$ such that $f^*(V)$ has a filtration: $f(V) = V_n \supset V_{n-1} \supset \ldots \supset V_0 = 0$ with $V_i/V_{i-1} \in \operatorname{Pic} Y$ for all i and V_i/V_{i-1} algebraically equivalent to V_f/V_{f-1} for all i and f.

(c) A vector bundle V on X is said to be *semistable with projective Chern classes zero* if $c(V)$, the total *Chern class* of V, is equal to $[1 + c_1(V)/n]^n$, where $n = rk\, V$, and V is semistable for some polarization of X.

Mukai [24] proved that (a) is equivalent to (b) and that (b) $\Rightarrow$ (c). That (c) $\Rightarrow$ (b) for abelian surfaces has been proved by Oda [30] and Takemoto [22]. We sketch a proof below that (b) $\Rightarrow$ (c) for abelian varieties of any dimension. So let V be semistable with $c(V) = [1 + c_1(V)/n]^n$. Let $n_X\colon X \to X$ be multiplication by n and put $L = c_1(V)$. Then $n_X^{\#}(L) \simeq L^{n^2} \otimes M$ for some M in $\operatorname{Pic}^0(X)$. Hence there exists $N \in \operatorname{Pic}(X)$ with $N \otimes n_X^{\#}(V)$ semistable and with zero Chern classes. By theorem 2, $n_X^{\#}(V) \otimes N$ has a filtration $n_X^{\#}(V) \otimes N = W_n \supset W_{n-1} \ldots \supset W_0 = 0$ with $W_i/W_{i-1} \in \operatorname{Pic}^0(X)$ for all i. It follows that $n_X^{\#}(V)$ has the desired filtration.

Mukai [24] has proved that a vector bundle V on an abelian variety X is stable and semi-homogeneous if and only if there exists an isogeny $f: Y \to X$ and a line bundle L on Y with $K(L) \cap \ker f = 0$ such that $V = f_*(L)$. However, neither f nor L is uniquely determined. It can be proved (Mehta and Nori [31]) that such vector bundles on X are in 1–1 correspondence with pairs (g, L), where $g: Z \to X$ is an isogeny, L is a line bundle on Z with $\ker g \subset K(L)$ and e^L/K non-degenerate.

We first prove theorems 1 and 2 in the case where k is the algebraic closure of a finite field. If X is a homogeneous space or an abelian variety then $H^0(X, T_X)$ generates T_X, where T_X is the tangent bundle of X. This implies that for any semistable sheaf V on X, $\pi^*(V)$ is also semistable, where $\pi: X \to X$ is the *Frobenius* morphism (cf. Proposition 1.1). We note that this is false unless T_X is generated by $H^0(X, T_X)$ (Gieseker [7]). Now if $X = G/P$ then we invoke [3] and [11, theorem 1.4] to assert that any V as in theorem 1 becomes trivial on some étale cover of X and hence on X itself as $\pi_1^{\text{alg}}(X) = 0$.

If X is an abelian variety, then we show that for any V as in theorem 2, there exists an integer n such that $n_X^*(V)$ is trivial on X, where $n_X: X \to X$ is multiplication by n. From this the existence of the desired filtration for V follows at once. The case where the ground field k is arbitrary is handled by the now standard method of reduction mod p. In § 1 we establish some notations and recall some known results and then prove theorem 1. Theorems 2 and 3 are proved in §2. "Stable" and "semistable" will always mean "χ-stable" and "χ-semistable" unless stated explicitly otherwise.

1. Semistable sheaves on homogeneous spaces

Let E be a torsion-free sheaf on X and define rational numbers $h_i(E)$ by

$$\chi_E(n) = \sum_{i=0}^{d} h_i(E) n^i,$$

and for any subsheaf F of E define numbers $\beta_i(F)$, in analogy with [12], by

$$\beta_i(F) = h_i(F)\,rk\,E - h_i(E)\,rk\,F, \quad 0 \leqslant i \leqslant d.$$

If $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence of subsheaves of F, then we have $\beta_i(F_2) = \beta_i(F_1) + \beta_i(F_3)$, $0 \leqslant i \leqslant d$.

The Hirzebruch-Riemann-Roch formula gives

$$\chi_E(n) = \frac{(\deg X\, rk\, E) n^d}{d!} + \frac{[c_1(E)\cdot H^{d-1} + (rk\cdot E/2) c_1(X)\cdot H^{d-1}] n^{d-1}}{d-1!}$$

+ terms of lower degree in n.

Hence we see that E is χ-semistable (resp. χ-stable) if and only if, for any subsheaf F of E, if $\beta_d(F) = \ldots \beta_{d-r+1}(F) = 0$ then $\beta_{d-r}(F) \leqslant 0$ (resp. < 0) for $1 \leqslant r \leqslant d$.

Now let E be a χ-unstable (*i.e.* not semistable) sheaf on X. Hence there exists $F \subset E$ with $\beta_d(F) = \ldots \beta_{d-r+1}(F) = 0$ but $\beta_{d-r}(F) > 0$. We may assume r to be *minimal* with respect to this property. We have two cases:

Case (*i*) $r = 1$: Then Langton [12] establishes the existence of a *unique* subsheaf B of E with the following properties: (i) E/B is torsion-free. (ii) $\beta_{d-1}(B) = \sup_{F \subset E} \{\beta_{d-1}(F)\}$. (iii) B is the subsheaf of smallest rank with properties 1 and 2. (iv) $\text{Hom}\ \mathbf{O}_X(B, E/B) = 0$.

Case (ii) $r > 1$: (This case has been treated in [17], but we recall the results for the sake of completeness). Let

$$S = \{G \subset E/\beta_d(G) = \ldots \beta_{d-r+1}(G) = 0\}.$$

As $r \geqslant 2$ and for any $G \in S$ we have $1 \leqslant rk\, G \leqslant rk\, E$, it follows from [8, 221–307] that S is a *bounded* family of sheaves on X. In particular we can define $\beta_{d-r} = \sup_{G \in S} \beta_{d-r}(G)$ and $T = \{G \in S/\beta_{d-r}(G) = \beta_{d-r}\}$. Let B be an element of T of smallest rank and define $\hat{B}$ *by*: $B \subset \hat{B} \subset E$ and $E/\hat{B}$ torsion-free. Then $\beta_d(\hat{B}) = 0$ (this is true for any subsheaf of E) and since $B \subset \hat{B}$ we have $h_{d-1}(B) \leqslant h_{d-1}(\hat{B})$, hence $\beta_{d-1}(\hat{B}) \geqslant 0$. If $\beta_{d-1}(\hat{B}) > 0$, then this contradicts the minimality of r. Hence $\beta_{d-1}(\hat{B}) = 0$. Similarly $\beta_{d-2}(\hat{B}) = \ldots \beta_{d-r+1}(\hat{B}) = 0$. So $\hat{B} \in S$. And as $\beta_{d-r}(B) \leqslant \beta_{d-r}(\hat{B})$, by maximality of $\beta_{d-r}(B)$ we get $\beta_{d-r}(B) = \beta_{d-r}(\hat{B})$. So we may assume $B = \hat{B}$ and hence we get that B is a subsheaf of E of *smallest* rank with $B \in S$ and E/B torsion-free and $\beta_{d-r}(B) = \beta_{d-r}$.

Further, we assert that B is the *unique* subsheaf of E of smallest rank with $B \in S$, E/B torsion-free and $\beta_{d-r}(B) = \beta_{d-r}$. For, let $F \subset E$ be another such. Then we claim that both $F \cap B$ and $F + B$ belong to S. Taking $F \cap B$ first, we have $\beta_d(F \cap B) = 0$ and $\beta_{d-1}(F \cap B) \leqslant 0$. If $\beta_{d-1}(F \cap B) < 0$, then in the isomorphism $B/F \cap B \simeq F + B/F$ we get $\beta_{d-1}(B/F \cap B) > 0$ and hence $\beta_{d-1}(F + B/F) > 0$ which in turn implies that $\beta_{d-1}(F + B) > 0$, a contradiction. Hence $\beta_{d-1}(F \cap B) = 0$. Similarly $\beta_{d-2}(F \cap B) = \ldots = \beta_{d-r+1}(F \cap B) = 0$ and also, by the same reasoning, we get $\beta_{d-1}(F + B) = \ldots = \beta_{d-r+1}(F + B) = 0$. Hence both $F \cap B$ and $F + B$ belong to S. Now we claim $\beta_{d-r}(F \cap B) = \beta_{d-r}(B)$. In any case we have $\beta_{d-r}(F \cap B) \leqslant \beta_{d-r}(B)$. Suppose strict inequality holds. Then, $\beta_{d-r}(B/F \cap B) > 0 \Rightarrow \beta_{d-r}(F + B/F) > 0 \Rightarrow \beta_{d-r}(F + B) > \beta_{d-r}(F) = \beta_{d-r}$, a contradiction. Now $rk\, B \cap F \leqslant rk\, B$ and so by the definition of B, $B \cap F = B$ or $F \supset B$. The uniqueness of B is hence established.

Now we can prove that $\mathrm{Hom}\ \mathbf{O}_X(B, E/B) = 0$. If not, let $f: B \to E/B$ be a map with image G/B and kernel K, where $B \subset G \subset E$. From the two exact sequences

$$0 \to B \to G \to G/B \to 0 \tag{1}$$

$$0 \to K \to B \to G/B \to 0 \tag{2}$$

we get

(1) $\beta_i(G) = \beta_i(G/B)$ for $d - r + 1 \leqslant i \leqslant d$

(2) $\beta_i(G/B) + \beta_i(K) = 0$ for $d - r + 1 \leqslant i \leqslant d$.

Hence $\beta_i(G) + \beta_i(K) = 0$ for $d - r + 1 \leqslant i \leqslant d$. But then we must have $\beta_i(G) = \beta_i(K) = 0$ for $d - r + 1 \leqslant i \leqslant d$, because if one of them, say $\beta_i(K) > 0$, then $\beta_d(K) = \beta_{d-1}(K) \ldots = \beta_{i+1}(K) = 0, \beta_i(K) > 0$, a contradiction. Hence both G and K belong to S. From (2) we get $2\beta_{d-r}(B) = \beta_{d-r}(K) + \beta_{d-r}(G)$. Hence at least one of $\beta_{d-r}(B)$ and $\beta_{d-r}(G)$ must be greater than or equal to $\beta_{d-r}(B)$. By our assumptions on B it now follows easily that $G = B$ and $f = 0$. In what follows we shall call B the β subsheaf of E.

Now we have

Proposition 1.1: Let V be a semistable torsion-free sheaf on X with X as in theorem 1 or theorem 2. Assume both X and V are defined over a finite field F_q and let $\pi: X \to X$ be the F_q-linear Frobenius. Then $\pi^*(V)$ is also semistable on X.

Proof: If $\pi^*(V)$ is not semistable on X then let B be the β subsheaf of V. Consider

$\mathrm{Der}_X(X_\pi, X_\pi)$, where $X_\pi = X$ is considered as a scheme over X via π. Every $D \in \mathrm{Der}_X(X_\pi, X_\pi)$ induces a $\mathbf{O}_X$-linear map $\overline{D}$: $\pi^\#(V) \to \pi^\#(V)$. We compose $\overline{D}$ with the projection $\pi^\#(V) \to \pi^\#(V)/B$. This induces a map $B \to \pi^\#(V)/B$ which can be seen to be $\mathbf{O}_{X\pi}$-linear. We have observed that such a map must be zero. Hence $\overline{D}$ takes B to B. Now the assumptions on X and [9, 190–23] imply that B descends to a subsheaf, say $\overline{B}$ of V. We have to show that $\overline{B}$ contradicts the χ-semistability of V and this follows from

Lemma 1.2: Let V be a torsion-free sheaf on a smooth projective variety X, both being defined over a finite field F_q. Let π: $X \to X$ be the F_q-linear Frobenius and define β_i for subsheaves of V and $\pi^\#(V)$ as before. Let $\overline{B} \subset V$ with $B = \pi^\#(\overline{B}) \subset \pi^\#(V)$ and assume that $\beta_d(B) = \ldots \beta_{d-r+1}(B) = 0$. Then $\beta_{d-r}(B) = q^r \beta_r(\overline{B})$.

Proof of Lemma 1.2: We use the *H-R-R* formula

$$\chi V(n) = [\mathrm{ch}\, V(n)\tau(X)]_d$$

where ch $V(n)$ is the *Chern character* of $V(n)$, $\tau(X)$ is the *Todd class* of X and $[\ \ ]_d$ denotes the homogeneous component of degree d. Let $\tau(X) \sum_{i=0}^{d} w_i$ and $\mathrm{ch}\, V = \sum_{i=0}^{d} l_i$. Then $\mathrm{ch}\, V(n) = \mathrm{ch}(V)\,\mathrm{ch}(H^n) = \left(\sum_{i=0}^{d} l_i\right)(1 + H^n + \ldots + (H^n)^d/d!)$. It follows that $h_k(V) = \sum_{i+j=d-k} (l_i w_j) H^k/k!$. Similarly, for $\overline{B} \subset V$, if $\mathrm{ch}(\overline{B}) = \sum_{i=0}^{d} t_i$ then $h_k(\overline{B}) = \left(\sum_{i+j=d-k} t_i w_j\right) H^k/k!$. Now $\mathrm{ch}(\pi^\#(V)) = \sum_{i=0}^{d} q^i l_i$ and $\mathrm{ch}(B) = \sum_{i=0}^{d} q^i t_i$. So $h_k(\pi^\#(V)) = \sum_{i+j=d-k} (q^i l_i w_j) H^k/k!$ and $h_k(B) = \sum_{i+j=d-k} (q^i t_i w_j) H^k/k!$. It now follows, after a short calculation, that if $\beta_d(B) = \ldots = \beta_{d-r+1}(B) = 0$ then $\beta_{d-r}(B) = q^r \beta_{d-r}(\overline{B})$, which completes the proof of Lemma 1.2 and hence that of Proposition 1.1. Q.E.D.

Remark 1.3: Proposition 1.1. in the case of $rk\,2$ sheaves on $\mathbf{P}^n$ has been proved by Barth [2].

We now take up the proof of theorem 1. We first assume that $k = \overline{F}_p$. Let X be as in theorem 1 and let $\mathcal{S} = \{V, r, c_1 \ldots, c_d\}$ be the set of all torsion-free semistable sheaves of $rk\,r$ on X with $c_i(V) = 0$, $1 \leqslant i \leqslant d$. By our assumption $(*)$ it follows that there exists a scheme T of finite type over k and a coherent torsion-free sheaf $\mathcal{V}$ on $X \times T$ flat over T such that if $V \in \mathcal{S}$ then there exists a closed point $t \in T$ with $\mathcal{V}_t \simeq V$. There also exists an integer n_0 such that for all $n \geqslant n_0$, we have

(i) $H^0(V(n))$ generates $V(n)$ for all $V \in \mathcal{S}$

(ii) $H^i(V(n)) = 0$ for all $i > 0$, for all $V \in \mathcal{S}$.

Fix an $n \geqslant n_0$. Let P be the Hilbert Polynomial of $V(n)$, $V \in \mathcal{S}$ and E be a trivial vector bundle on X of rank $= \dim H^0(V(n))$, $V \in \mathcal{S}$. Let $Q = Q(E/P)$ be the *Quot* scheme of coherent quotients of E with Hilbert polynomial P. For any $V \in \mathcal{S}$, we may assume the existence of a finite extension $F_q \supset F_p$ with X, Q and V all being defined over F_q, which gives us a F_q-rational point of Q. As Q has only finitely many F_q-rational points, we get the important consequence that there are only finitely many isomorphism classes (over k) of elements of $\mathcal{S}$ which are defined over F_q. Let π: $X \to X$ be the F_q-linear Frobenius, we get $\pi^\#(V) \in \mathcal{S}$ by proposition 1.1. Hence we infer the existence of an infinite sequence of integers $(n_k)_{k=1}^{\infty}$ with $(\pi^{n_i})(V) \simeq (\pi^{n_j})(V)$. Now we deduce V is locally free:

Lemma 1.4: Let M be a finitely generated module over a regular local ring in char $p > 0$ and assume that $\pi^{\#}(M) \simeq M$. Then M is free.

Proof: Let

$$0 \to F_d \xrightarrow{\phi_d} F_{d-1} \ldots F_1 \xrightarrow{\phi_1} F_0 \to M \to 0$$

be a minimal resolution of M. Then by [20, theorem 1.13], the minimal resolution of $\pi^{\#}(M)$ is

$$0 \to F_d \xrightarrow{\phi_d^q} F_{d-1} \ldots F_1 \xrightarrow{\phi_1^q} F_0 \to \pi^{\#}(M) \to 0$$

If $M \simeq \pi^{\#}(M)$ then these two complexes are also isomorphic and hence it follows by comparing their minors for instance, that M is free.

Q.E.D.

So now each $V \in \mathcal{S}$ is a vector bundle on X and there exist integers m, n with $m > n$ and $(\pi^m)^{\#}(V) \simeq (\pi^n)^{\#}(V)$. Putting $(\pi^n)^{\#}(V) = \overline{V}$ and $m - n = t$, we get $(\pi^t)^{\#}(\overline{V}) \simeq \overline{V}$. Now we quote

Proposition 1.5: (Lange and Stuhler [11, theorem 1.4]) Let V be a vector bundle on a scheme X in char $p > 0$. Then $(\pi^t)^{\#}(V)$ is isomorphic to V for some t if and only if V becomes trivial on an étale cover of X.

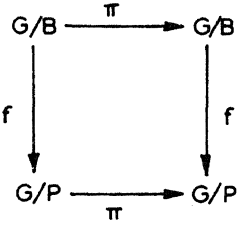
So it follows that $\overline{V}$ is trivial on an étale cover of X.

Proposition 1.6: Let $X = G/P$ be a homogeneous space in char $p > 0$ with G reductive and P parabolic. Let V be a vector bundle on X such that $(\pi^n)^{\#}(V)$ is trivial on X. Then V itself is trivial.

Proof: We first assume P is a Borel subgroup of G. Then by [3] there is a sequence

$$Z = Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} Z_2 \ldots \to Z_m = \mathbf{P}^1$$

with all the f_i locally trivial $\mathbf{P}^1$ fibration and $f: Z \to X$ a birational morphism. Let $g_i: Z \to Z_i$ be the projection and assume that we have proved that $g_i^{\#}(W) \simeq f^{\#}(V)$ for some W on Z_i. It is clear by induction that for any $\mathbf{P}^1 \subset Z_i$, there is a section $s: \mathbf{P}^1 \to Z$ so that $g_i \circ s$ is the inclusion of $\mathbf{P}^1$ in Z_i. Consequently $W \mid \mathbf{P}^1 \simeq s^{\#} f^{\#}(V)$ and this bundle has the property that a high power of its Frobenius is étale-trivialisable, and is therefore easily seen to be trivial by writing it as a direct sum of line bundles. In particular W is trivial when restricted to all the fibres of f_i, showing that $W \simeq f_i^{\#} f_{i\#}(W)$. Putting $f_{i\#}(W) = W_1$, we find that $g_{i+1}(W_1) \simeq f^{\#}(V)$. By induction, $f^{\#}(V)$ is trivial, and because $f_{\#} f(V)^{\#} V$, V itself is trivial. Now assume that P is an arbitrary parabolic subgroup of G and V is a vector bundle on G/P such that $(\pi^n)^{\#}(V)$ is trivial. Let $B \subset P$ be a Borel subgroup of G and let $f: G/B \to G/P$ be the canonical map. We have a commutative diagram:



If $(\pi^n)^*(V)$ is trivial on G/P, then $f^*(\pi^n)^*(V) = (\pi^n)^* f^*(V)$ is trivial on G/B. So $f^*(V)$ is a bundle on G/B with $(\pi^n)^* f^*(V)$ trivial on G/B. By the first part of this proposition, it follows that $f^*(V)$ is trivial on G/B and since V is isomorphic to $f_* f^*(V)$, V is trivial on G/P.

Now suppose $X = G/P$ is defined over an arbitrary algebraically closed field k and V is a semistable torsion-free sheaf on X with $c_i(V) = 0$, $1 \leqslant i \leqslant d$. We may assume (i) There exists a subring A of k which is finitely generated over $\mathbf{Z}$, with quotient field K, such that X, V and G are defined over K. (ii) There is a smooth reductive group-scheme $G_A \to A$ whose general fibre is G. (iii) $P_A \subset G_A$ is a parabolic subgroup scheme with quotient $G_A | P_A$ denoted by X_A. (iv) There is a sheaf V_A on X_A flat over A with $V_A \otimes_A (\Omega)$ semistable and torsion-free on $X_A \otimes_A (\Omega)$ for every geometric point Ω of A and $c_i(V_A \otimes_A \Omega) = 0$, $1 \leqslant i \leqslant d$ and $V_A \otimes_A K = V$.

Now for every maximal ideal $\mathbf{m}$ of A, $F_q = A/\mathbf{m}$ is a finite field and by propositions 1.1–1.6, $V_A \otimes_A \overline{F}_q$ is a trivial vector bundle on $X_A \otimes_A \overline{F}_q$. It easily follows that V_K is trivial on X_K and hence V is trivial on X. Q.E.D.

If we work instead with μ-stability and μ-semistability we have the following improvement of proposition 1.1:

Proposition 1.7: Let X be a projective space or a Grassmannian in char $p > 0$ and let V be a μ-stable sheaf on X. Then $\pi^*(V)$ is also μ-stable.

Proof: In any case, by the proof of proposition 1.1 we infer that $\pi^*(V)$ is μ-semistable. According to [26, theorem 3.23], for any smooth variety X in char $p > 0$ with $\pi^*(V)$ μ-semistable whenever V is μ-semistable, we have that $V_1 \otimes V_2$ is μ-semistable if both V_1 and V_2 are μ-semistable. Now we prove that $\pi^*(V)$ is μ-stable. Let W be a subsheaf of $\pi^*(V)$ with $\mu(W) = \mu(\pi^*(V))$. Now if T_X is the tangent bundle of X, we get a $\mathbf{O}_X$-linear homomorphism $f: T_X \to Hom(W, \pi^*(V)/W)$. If X is a projective space or a Grassmannian, then T_X is μ-semistable with $\mu(T_X) > 0$, as can easily be checked (see below and [12]). But $Hom(W, \pi^*(V)/W)$ is μ-semistable and of degree zero and hence $f = 0$. Hence W descends to a subsheaf $\overline{W}$ say, of V with $\mu(\overline{W}) = \mu(V)$, contradicting the μ-stability of V.

Remark: The tangent bundle of the Grassmanian is a homogeneous bundle, from which it follows that each member of the Harder–Narasimhan flag is also homogeneous. Now the Grassmanian is $Sl(n)/P$ and the action of P on Lie $(Sl(n))$/Lie (P) is irreducible, and therefore there are no proper non-zero homogeneous subbundles of the tangent bundle. Thus the tangent bundle is semi-stable. Note that its μ is positive because its determinant is in fact very ample.

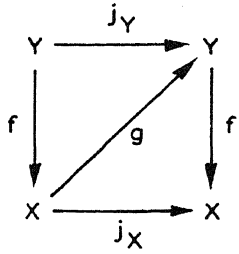
2. Semistable sheaves on abelian varieties

We now turn to the proof of theorem 2. X is an abelian variety over an algebraically closed field k and V is a semistable torsion-free sheaf on X with $c_i(V) = 0, 1 \leqslant i \leqslant d$. We first assume that $k = \overline{F}_p$. Now propositions 1.1, 1.2, 1.4 and 1.5 remain valid for X and V and so we have

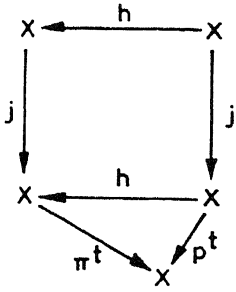
(i) V is a *vector bundle* on X.

(ii) $(\pi^t)^*(\overline{V}) \simeq \overline{V}$, where $\overline{V} = (\pi^n)^*(V)$ and m and n are such that $(\pi^m)^*(V) \simeq (\pi^n)^*(V)$.

Again by [11, theorem 1.4] there exists an étale covering $f: Y \to X$ with $f^{\#}(\overline{V})$ trivial on Y. By [18, §18] Y is an abelian variety, f is a homomorphism and there is a commutative diagram



where j_Y and j_X are multiplication by j on X and Y respectively. Hence $(j_X)^{\#}(\overline{V})$ is trivial on X. Now multiplication by $p^t: X \to X$ ($p = \text{char } k$) induces the zero map on the tangent spaces at all the points of X hence there is a commutative diagram



where $h: X \to X$ is some homomorphism. We therefore get that $m^{\#}(V)$ is trivial on X, where $m = p^t j$. Let X_m be the *group-scheme* kernel of $m: X \to X$. Then $m: X \to X$ is a *principal bundle* with structure group X_m. As $m^{\#}(V)$ is trivial on X, V is the bundle on X associated to a *linear representation* of X_m on a finite dimensional vector space say W. As X_m is a commutative group scheme there is a filtration on W

$$0 = W_0 \subset W_1 \ldots \subset W_r = W$$

with each W_i a X_m-subspace of W and $\dim W_{i+1}/W_i = 1$ for $0 \leqslant i \leqslant r-1$. It follows that V has a filtration

$$0 = V_0 \subset V_1 \ldots \subset V_r = V$$

with each V_{i+1}/V_i a line bundle on X, say L_{i+1}. Now $m^{\#}(L_{i+1})$ is trivial on X for all i and hence by [18, page 75] each L_{i+1} is algebraically equivalent to zero, *i.e.* each $L_{i+1} \in \text{Pic}^0(X)$.

Now assume k to be arbitrary. As in § 1 we may assume (i) There exists a subfield K of k which is finitely generated over the prime field such that both X and V are defined over K. (ii) There exists a Noetherian subring A of K with K the quotient field of A such that there is an *abelian scheme* X_A over A and a sheaf V_A on X_A flat over A with general fibres X and V respectively and $V_A \times_A \Omega$ semistable torsion-free with zero Chern classes for every geometric point Ω of A.

Let $f: X_A \to A$ be the structural map and let $\hat{f}: \hat{X}_A \to A$, where $\hat{X}_A$ is the dual of X_A. Let L be a Poincare bundle on $X_A \otimes_A X_A$. Consider $W = Hom[L, p_1^{\#}(V_A)]$ on $X_A x_A X_A$ where $p_1 : X_A x_A \hat{X}_A \to X_A$ is the first projection. Define $D = (t \in \hat{X}_A | H^0(W_t) \geqslant 1)$. D is a closed subset of X_A. Giving it to the reduced structure we have

Proposition 2.1: The restriction of $\hat{f}$ to D is a surjection: $D \to A$.

Proof: If $A - \hat{f}(D) \neq \phi$ then there exists $g \in A$ and a maximal ideal $\mathbf{m}$ with $g\mathbf{m} \neq 0$. But for any such $\mathbf{m}$, $A/\mathbf{m}$ is a finite field, which contradicts what has been proved earlier. Hence $f: \hat{D} \to A$ is surjective Q.E.D.

So after restricting A suitably we may assume that $D \to A$ and hence $\hat{X}_A \to A$ has a section $\sigma: A \to \hat{X}_A$. We get a line bundle $M = (id_X \times \sigma)^{\#}(L)$ on X and we consider $Hom(M, V)$ on X. We have $H^0[Hom(M_x, V_x)] \geqslant 1$ for every closed point x of A and hence for every point of A. Again by restricting A if necessary we may assume $H^0[Hom(M_x, V_x)] = 1$ for every $x \in A$. Then $f_{\#}[Hom(M, V)]$ is a line bundle on A and we consider the composite

$$M \otimes f^{\#} f_{\#}[Hom(M, V)] \to [M \otimes Hom(M, V)] \to V$$

which we know to be an inclusion of bundles for every closed point of A. Hence M is a line subbundle of V, at least up to a line bundle coming from A. Now $P_M(n) = P_V(n)$ and both M and V are semistable, hence V/M is semistable with zero Chern classes. By induction V/M has a filtration

$$0 = V_1 \subset V_2 \ldots \subset V_r = V/M$$

where each $V_i/V_{i-1} \in \mathrm{Pic}^0(X), 1 \leqslant i \leqslant r$. Hence V has a filtration

$$0 = V_0 \subset V_1 = M \subset V_2 \ldots \subset V_r = V \text{ with}$$

$$V_i/V_{i-1} \in \mathrm{Pic}^0(X),\ 1 \leqslant i \leqslant r. \qquad \text{Q.E.D.}$$

Now we can deduce theorem 3 from theorems 1 and 2. We are given a smooth surjective map $f: G \to X$ where G is a smooth algebraic group. Now there exists a *maximal* linear subgroup L of G and an exact sequence.

$$0 \to L \to G \to G/L \to 0$$

with G/L being an abelian variety. We also get a map $G/H \to G/L + H$ with all fibres isomorphic to $L/L \cap H$, and a map $G/L \to G/L + H$ with fibres $L + H/L$. So finally there is a map, also denoted by f, from X to $G/L + H$, which is an abelian variety, which we denote by A. We also have $f^{-1}(a) \simeq L/L \cap H$ for all $a \in A$.

Let V be a semistable torsion free sheaf on X with $c_i(V) = 0, 0 \leqslant i \leqslant d$. Then, by what has been proved earlier, we get

(i) V is a vector bundle on X,

(ii) $(\pi^n)^{\#}(V) \simeq (\pi^n)^{\#}(V)$.

Then $(\pi^m)^{\#}(V)/f^{-1}(a) \simeq (\pi^n)^{\#}(V)/f^{-1}(a)$ for all $a \in A$ and hence $V/f^{-1}(a)$ is trivial for all $a \in A$.

Hence $V = f^{-1}(W)$ with W being a vector bundle on A. Put $\overline{V} = (\pi^t)^{\#}(V)$ and $\overline{W} = (\pi^t)^{\#}(W)$ where $t = m - n$. Then $(\pi^t)^{\#}(\overline{V}) \simeq \overline{V}$ and $(\pi^t)^{\#}(\overline{W})$ is a bundle on

A with $f^*(\pi^t)^*(\overline{W}) = (\pi^t \circ f)^*(\overline{W}) = (\pi^t)^* f^*(\overline{W}) = (\pi^t)^*(\overline{V}) = \overline{V} = f^*(\overline{W})$. So $(\pi^t)^*(\overline{W}) \simeq \overline{W}$ on A and hence $m^*(W)$ is trivial on A for some integer m. It follows that W has a filtration.

$$0 = W_0 \subset W_1 \ldots \subset W_r = W$$

with $W_i/W_{i-1} \in \mathrm{Pic}^0(A)$, $1 \leqslant i \leqslant r$. Hence V has a filtration

$$0 = V_0 \subset V_1 \ldots \subset V_r = V$$

with V_i/V_{i-1} $\mathrm{Pic}^0(X)$, $1 \leqslant i \leqslant r$. Q.E.D.

Remark 2.2: Theorem 2 generalizes a theorem of Atiyah [1, theorem 5] in the case where $\dim X = 1$.

Remark 2.3: Let V be a semistable sheaf on $\mathbf{P}^n$ in *Characteristic zero.* Then Spindler [21, theorem 3.2] has proved that the instability degree of the restriction of V to a general $\mathbf{P}^1 \subset \mathbf{P}^n$ is bounded above only by the rank of V and not by the Chern classes of V. Theorem 1 shows this result to be false in char p. Take a semistable V on $\mathbf{P}^n$ with $V/\mathbf{P}^1$ not semistable for a general $\mathbf{P}^1 \subset \mathbf{P}^n$. Then $(\pi^t)^*(V)$ is semistable on $\mathbf{P}^n$ for all t but the instability degree of $(\pi^t)^*(V)/\mathbf{P}^1$ goes to infinity.

Remark 2.4: Ellencwajg and Forster [4] have proved the following

Take the set S of semistable sheaves on $\mathbf{P}^n$ in characteristic zero with a fixed rank and c_1 and c_2. Then S is *bounded.* If the same result is true in char p one gets an interesting consequence: take S with $c_1 = c_2 = 0$, Then $(\pi^t)^*(V) \in S$ for all $V \in S$ and $t > 0$ hence every $V \in S$ must become trivial on every $\mathbf{P}^1 \subset \mathbf{P}^n$. This easily implies that each $V \in S$ must be trivial on $\mathbf{P}^n$.

Remark 2.5: Assume V is a vector bundle on an abelian variety A such that V has a filtration

$$0 = V_0 \subset V_1 \ldots \subset V_r = V$$

with $V_i/V_{i-1} \in \mathrm{Pic}^0(A)$, $1 \leqslant i \leqslant r$. We can classify such V as follows

(i) Let $V = \bigoplus_{i=1}^{k} W_i$, with each W_i indecomposable. Then for each i, $1 \leqslant i \leqslant k$, there is a *unique* line bundle $L_i \in \mathrm{Pic}^0(A)$, such that $L_i \otimes W_i$ has a filtration by subbundles with each successive quotient *trivial.* (cf. [19], Chp. 4, [23] Thm. 2.3). (ii) Vector bundles V on A admitting filtrations with trivial quotients as in (i) are in canonical 1–1 correspondence with the isomorphism classes of finite length modules over the local ring at the identity of the dual abelian variety $\hat{A}$. (cf. [19], Chp. 4, [24] Thm. 4.19, [25] Ex. 2.9).

Remark 2.6: Let X be a homogeneous space or an abelian variety. Then it is easy to see that all the results of this paper remain valid for *μ-semistable* torsion-free sheaves on X if one makes the following assumption:

The set V of all μ-semistable sheaves on X is *bounded* for fixed values of $rk\, V$ and $c_i(V)$, $0 \leqslant i \leqslant \dim X$.

S Ramanan has pointed out to us the following application of theorem 2.

Proposition 2.7: Let V be a stable vector bundle on an abelian variety X of dim 3 in char zero. Assume that $NS(X) \simeq \mathbf{Z}$. If Bogomolov's inequality is not strict, then there exists an isogeny $f\colon Y \to X$ and a line bundle L on Y with $f_*(L) \simeq V$.

Proof: By [26, theorem 3.18] End V is semistable. By our assumption on $NS(X)$, $c_2(\text{End } V) = \alpha H^2$ in $H^4(X; Q)$ for some rational number. α and Bogomolov's inequality says that $c_2(\text{End } V).H \geqslant 0$ (here H denotes the polarization on X). If the inequality is not strict, then $c_2(\text{End } V) = 0$. In any case, $c_i(\text{End } V) = 0$ for odd i.

Hence it follows from theorem 2 that $E \in \text{Pic}^0(X)$ with $E \subset Ad\, V \subset \text{End } V$ inducing $\phi: E \otimes V \to V$. As V is stable, ϕ is an isomorphism and as $E \subset Ad\, V$, E is not the trivial line bundle. By taking determinants we get $E^n \simeq \mathbf{O}_X$, where $n = rk\, V$. Now let $f: Y \to X$ be the isogeny of least degree such that $f^*(E)$ is trivial. It is easy to see that there is a bundle W on Y with $f_*(W) \simeq V$ (See, for example, [32], Lemma 2.5). Repeating the above argument for (Y, W) in place of (X, V), the result follows. Note that the entire argument is valid in char p is one knew Bogomolov's inequalities for an abelian threefold in char p.

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Thermal instability of a horizontal layer of micropolar fluid with heat source

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Abstract. The thermal instability of a horizontal layer of micropolar fluid which loses heat throughout its volume at a constant rate has been considered. The influence of the various micropolar fluid parameters on the onset of convection have been analysed. It is found that heat source and heat sink have the same destabilising effect in micropolar fluid. It is observed that the horizontal dimension of the cells remains insensitive to the changes in the micropolar fluid parameters and also to the heat source parameter Q except for Q values near zero, where the change is drastic. Further, it is observed that though the vertical component of velocity and the curl of microrotation do not vanish anywhere between the two boundaries for $Q = 0$, they vanish at a point nearer to the lower boundary even for a small change in the Q value.

Keywords. Free convection; stability; micropolar fluid; heat source.

1. Introduction

Although a number of studies on stability of Newtonian and non-Newtonian fluids have been reported in literature, these do not give satisfactory results if the fluid is a mixture of heterogeneous means such as liquid crystals, ferro liquid, liquid with polymeric additives, etc., which is more realistic and important from the technological point of view. For the realistic description of the flow of such rheologically complex fluids there exist several theories, *e.g.*, polar fluids, dipolar fluids, couple stress fluids, etc. However, it has been demonstrated by Ariman *et al* [2] that for linear and viscous fluids all these theories can be considered as equivalent to micropolar fluid theory [7]. This theory is capable of explaining the behaviour of fluids which exhibit certain microscopic effects arising from the local structure and micromotions of fluid elements, such as real fluids with suspensions and colloidal fluids. Several possible applications of micropolar fluid theory to describe the non-Newtonian behaviour of certain real fluids have been indicated by Ariman *et al* [3] and Eringen [8].

The work of stability of micropolar fluid is relatively of recent origin. The effects of microstructure in the thermal instability have been studied by Ahmadi [1], Bhattacharyya and Jena [4], Datta and Sastry [6], and Lebon and Perez-Garcia [9]. These studies are concerned with an initial quiescent state in which the fluid temperature decreases linearly with height. However, consideration of actual physical situation suggests that the above condition of linearly varying fluid temperature in the quiescent state may be too restrictive. In many physical problems, for example, convective studies with internal heat sources proposed for the earth's mantle, radiative heat transfer, heat generating fluids, fluids with chemically reacting constituents, the rising of volcanic liquid with bubbles, etc., there is always a thermal dissipation inside

the fluid even when it is at rest. In such a situation it is more appropriate to consider the temperature profile to be non-linear. Therefore, it is of interest to determine in what way the stability would be affected if the fluid possesses a non-linear temperature profile in the quiescent state.

In the present paper, we have investigated the effect of a uniform distribution of heat source, which gives rise to a non-linear temperature profile in the quiescent state, on the stability of a horizontal layer of micropolar fluid heated from below. In §2, we have given the formulation of the characteristic value problem for thermal instability. Using the variational method, the characteristic value problem has been solved in §3. It has been established in §4 that heat source and heat sink have the same effect in micropolar fluid. Finally, in §5, we have presented the results.

2. Formulation of the problem

We consider an infinite horizontal layer of an incompressible micropolar fluid of finite depth d having no body couple. A coordinate system $0x_1x_2x_3$ is chosen having $0x_3$ in the vertical plane and $0x_1$, $0x_2$ in the horizontal plane. The fluid is heated from below and the lower and upper boundaries are kept at constant temperatures T_0 and T_1 respectively. It is assumed that the heat is lost at a constant rate throughout the volume of the fluid. In accordance with free convection flow, the density has been considered a variable only in forming the buoyant force.

In an initial quiescent state, heat transport is by conduction alone and the temperature and the pressure satisfy the equation [6]

$$p_{,k} = \rho f_k, \tag{1}$$

$$T_{,kk} = -\frac{h}{k_c c_v}, \tag{2}$$

where $p, T, \rho, f_k, h, k_c, c_v$ are respectively the hydrostatic pressure, absolute temperature, constant mass density, body force vector, constant heat source per unit mass, thermal diffusivity and specific heat of fluid at constant volume. In (1) and (2), an index following a comma indicates the partial derivative with respect to the spatial rectangular coordinates x_k.

Assuming temperature and pressure to depend only on x_3, from (1) and (2), we have

$$\frac{\partial p}{\partial x_3} = -\rho g, \tag{3}$$

$$T = T_0 - \frac{\Delta T}{d} x_3 - \frac{h x_3}{2k_c c_v}(x_3 - d), \tag{4}$$

where g is the acceleration due to gravity and $\Delta T = T_0 - T_1$.

For small temperature difference we can have an equation of state as

$$\rho = \rho_0[1 - \lambda(T - T_0)], \tag{5}$$

where λ is the coefficient of volume expansion and $\rho = \rho_0$ when $T = T_0$.

Let v_i, ν_i, p' and T' be the perturbations in velocity, microrotation, pressure and

temperature respectively. Then under the framework of Boussinesq's approximation, the linearized equations governing the disturbances [6] can be written, making use of (3)–(5), as

$$\frac{\partial v_i}{\partial x_i} = 0, \tag{6}$$

$$\rho_0 \frac{\partial v_i}{\partial t} = \rho_0 g \lambda T' \delta_i - \frac{\partial p'}{\partial x_i} + (\mu_v + k_v)\nabla^2 v_i + k_v \Omega_i, \tag{7}$$

$$\rho_0 j \frac{\partial v_i}{\partial t} = (\alpha_v + \beta_v)\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right) + \gamma_v \nabla^2 v_i + k_v(\omega_i - 2v_i), \tag{8}$$

$$\left(\frac{\partial}{\partial t} - k_c \nabla^2\right) T' = \left[\frac{\Delta T}{\mathrm{d}} + \frac{h}{k_c c_v}\left(x_3 - \frac{d}{2}\right)\right] \cdot \left(v_3 - \frac{\alpha \Omega_3}{\rho_0 c_v}\right). \tag{9}$$

In the above equations, μ_v, k_v, j, α_v, β_v, γ_v and α are the material parameters characterizing the micropolar property of the fluid. e_{ilm} is the alternating tensor, t is the time, $\delta_i = (0, 0, 1)$,

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Omega_i = e_{ilm}\frac{\partial v_m}{\partial x_1}, \quad \omega_i = e_{ilm}\frac{\partial v_m}{\partial x_1}.$$

In accordance with the linear stability theory, assuming the changes in velocity, microrotation and temperature to be small, we have neglected all the small quantities of order higher than the first in (6)–(9) which includes viscous dissipation also. Equations (6)–(9) reduce to the corresponding equations for Newtonian fluids [12], [13] by setting $j = k_v = \alpha_v = \beta_v = \gamma_v = \alpha = 0$.

2.1 *Boundary conditions*

Case (*a*): When both the boundaries are free, we have

$$v_3 = 0, \quad \frac{\partial^2 v_3}{\partial x_3^2} = 0, \quad \Omega_3 = 0, \quad T' = 0 \text{ at } x_3 = 0, \mathrm{d}. \tag{10}$$

Case (*b*): When both the boundaries are rigid, we have

$$v_3 = 0, \quad \frac{\partial v_3}{\partial x_3} = 0, \quad \Omega_3 = 0, \quad T' = 0 \text{ at } x_3 = 0, \mathrm{d}. \tag{11}$$

It may be noted here that the matter of proper boundary condition for microrotation on the free surface is as yet unresolved. At present, there seems to be no physical theory that enables us to choose the appropriate boundary conditions on the microrotation. Hence, it is reasonable, under such circumstances, to choose the simplest boundary condition for microrotation. Most commonly used boundary conditions for microrotation are the assumption of no spin condition, *i.e.* the microrotation to be zero on the surface. The boundary condition for microrotation presented in (10) and (11) follows from the no spin condition for microrotation [1]. The same type of boundary conditions for microrotation have also been employed in [4], [6] and [9]–[11].

We now introduce the following non-dimensional parameters:

$$(x, y, z) = \left(\frac{x_1}{d}, \frac{x_2}{d}, \frac{x_3}{d}\right), t_1 = \frac{(2\mu_v + k_v)t}{\rho_0 d^2}, w = \frac{v_3}{(k_c/d)},$$

$$\theta = \frac{T'}{\Delta T}, \quad \Omega = \frac{\Omega_3}{(k_c/d^3)}, \quad N_1 = \frac{k_v}{(2\mu_v + k_v)}, \quad N_2 = \frac{j}{d^2},$$

$$N_3 = \frac{\gamma_v}{(2\mu_v + k_v)d^2}, \quad N_5 = \frac{\alpha}{\rho_0 c_v d^2}, \quad \mathrm{Pr} = \frac{(2\mu_v + k_v)}{\rho_0 k_c},$$

$$Q = \frac{hd^2}{k_c c_v (\Delta T)}, \quad \mathrm{Ra} = \frac{\rho_0 g \lambda (\Delta T) d^3}{(2\mu_v + k_v) k_c}. \tag{12}$$

In (12), Pr, Q and Ra are respectively the Prandtl number, the heat source parameter and the Rayleigh number. N_1, N_2, N_3 and N_5 are the micropolar fluid parameters characterizing vortex viscosity, microinertia density, spin-gradient viscosity and micropolar heat conduction respectively.

In accordance with the usual normal mode analysis, we assume that the perturbations w, Ω and θ have the form

$$[w, \Omega, \theta] = [W(z), G(z), \Theta(z)] \cdot \exp[i(a_1 x + a_2 y) + \sigma t_1], \tag{13}$$

where σ is the nondimensional frequency parameter and a is the nondimensional wave number characterized by $a^2 = \hat{a}_1^2 + \hat{a}_2^2$.

Proceeding in a similar manner as given by Chandrasekhar [5] and making use of (12) and (13), we obtain, from (6)–(9), the equations governing the disturbances in nondimensional form as

$$(D^2 - a^2)\left[\tfrac{1}{2}(1 + N_1)(D^2 - a^2) - \sigma\right] W + N_1 (D^2 - a^2) G - \mathrm{Ra}\, a^2 \Theta = 0, \tag{14}$$

$$[N_3 (D^2 - a^2) - 2N_1 - N_2 \sigma] G - N_1 (D^2 - a^2) W = 0, \tag{15}$$

$$(D^2 - a^2 - \mathrm{Pr}\,\sigma)\Theta + (W - N_5 G)[1 + Q(z - \tfrac{1}{2})] = 0, \tag{16}$$

where $D = \dfrac{d}{dz}$.

The boundary conditions (10) and (11) now become

$$W = 0, \quad D^2 W = 0, \quad G = 0, \quad \Theta = 0 \text{ at } z = 0 \text{ and } 1, \tag{17}$$

$$W = 0, \quad DW = 0, \quad G = 0, \quad \Theta = 0 \text{ at } z = 0 \text{ and } 1. \tag{18}$$

Equations (14)–(16) together with the boundary conditions (17) or (18) constitute the characteristic value problem for thermal instability.

3. Solution of the characteristic value problem

In the present analysis, we confine our attention to the convective marginal state by setting $\sigma = 0$ in (14)–(16).

Let us assume that the solutions of (14)–(16) with the boundary conditions (17) and (18) are of the form

$$W = \sum_{n=1}^{\infty} A_n W_n, \quad G = \sum_{n=1}^{\infty} A_n G_n, \quad \Theta = \sum_{n=1}^{\infty} A_n \sin n\pi z, \tag{19}$$

where A_n's are constants.

Substituting (19) into (14) and (15) and solving these equations for W_n and G_n, we obtain

$$W_n = c_n \gamma_n \mathrm{Ra}\, a^2 \sin n\pi z + \sum_{j=1}^{2} (A_j^{(n)} \cosh q_j z + B_j^{(n)} \sinh q_j z) + A_3^{(n)} z \cosh q_2 z + B_3^{(n)} z \sinh q_2 z, \tag{20}$$

$$G_n = N_1 \gamma_n \mathrm{Ra}\, a^2 (n^2\pi^2 + a^2) \sin n\pi z - \frac{(N_1+2)}{N_3} [A_1^{(n)} \cosh q_1 z + B_1^{(n)} \sinh q_1 z] - A_3^{(n)} q_2 \sinh q_2 z - B_3^{(n)} q_2 \cosh q_2 z + C_1^{(n)} \cosh q_3 z + C_2^{(n)} \sinh q_3 z, \tag{21}$$

where

$$q_1^2 = a^2 + \frac{N_1(N_1+2)}{N_3(N_1+1)}, \quad q_2^2 = a^2, \quad q_3^2 = a^2 + \frac{2N_1}{N_3},$$

$$c_n = N_3(n^2\pi^2 + a^2) + 2N_1,$$

$$n = 1/[(n^2\pi^2 + a^2)^2 \{(1+N_1)N_3(n^2\pi^2 + a^2) + 2N_1(1+N_1) - N_1^2\}],$$

the $A_j^{(n)}$'s, $B_j^{(n)}$'s ($j = 1, 2, 3$) and the $C_j^{(n)}$'s ($j = 1, 2$) are constants of integration. Substituting (19) into (16), we obtain

$$\sum_{n=1}^{\infty} A_n (n^2\pi^2 + a^2) \sin n\pi z = \sum_{n=1}^{\infty} A_n (W_n - N_5 G_n)[1 + Q(z - \tfrac{1}{2})], \tag{22}$$

Multiplying both sides of (22) by $\sin m\pi z$ and integrating from 0 to 1, we get

$$\sum_{n=1}^{\infty} \left[K_{mn} - \frac{(n^2\pi^2 + a^2)}{2} \delta_{mn} \right] A_n = 0, \tag{23}$$

where

$$K_{mn} = \int_0^1 (W_n - N_5 G_n)[1 + Q(z - \tfrac{1}{2})] \sin m\pi z \, dz. \tag{24}$$

and δ_{mn} is the Kronecker delta.

The secular determinant giving the eigenvalue of Ra is obtained as

$$\left| \left| K_{mn} - \frac{(n^2\pi^2 + a^2)}{2} \delta_{mn} \right| \right| = 0. \tag{25}$$

Case (a)—Free boundaries:

Using (14) and (15) and the boundary conditions (17), we get

$$W_n = D^2 W_n = D^4 W_n = G_n = 0 \text{ for } z = 0 \text{ and } 1. \tag{26}$$

Using (20) and (21) and the boundary conditions (26), we obtain

$$A_1^{(n)} = A_2^{(n)} = A_3^{(n)} = B_1^{(n)} = B_2^{(n)} = B_3^{(n)} = C_1^{(n)} = C_2^{(n)} = 0.$$

Hence the expressions for W_n and G_n in (20) and (21), respectively reduce to

$$W_n = c_n \gamma_n \, \mathrm{Ra}\, a^2 \sin n\pi z, \tag{27}$$

$$G_n = N_1 \gamma_n \, \mathrm{Ra}\, (n^2\pi^2 + a^2) \sin n\pi z. \tag{28}$$

Substituting (27) and (28) into (24), we get

$$K_{mn} = \begin{cases} \dfrac{2mn \, \mathrm{Ra}\, a^2 \gamma_n Q[(-1)^{m+n} - 1] \cdot [c_n - N_5 N_1 (n^2\pi^2 + a^2)]}{\pi^2 (m^2 - n^2)^2}, \\ \qquad\qquad\qquad\qquad \text{when } m \neq n, \\ \frac{1}{2}\gamma_m \, \mathrm{Ra}\, a^2 [c_m - N_5 N_1 (m^2\pi^2 + a^2)], \quad \text{when } m = n. \end{cases} \tag{29}$$

Case (b)—Rigid boundaries:

Using (14) and (15) and the boundary conditions (18), we get

$$W_n = DW_n = G_n = 0 \quad \text{for} \quad z = 0 \text{ and } 1. \tag{30}$$

Moreover,

$$[(D^2 - a^2)\{N_3(1 + N_1)(D^2 - a^2) + N_1^2\}] W_n = 0 \text{ for } z = 0 \text{ and } 1. \tag{31}$$

Using (20) and (21) and the boundary conditions (30) and (31), we obtain, after some simplifications,

$$[B_1^{(n)}, B_2^{(n)}, B_3^{(n)}, A_1^{(n)}, A_2^{(n)}, A_3^{(n)}, C_1^{(n)}, C_2^{(n)}] = [X_1^{(n)}, X_2^{(n)}, X_3^{(n)}, X_4^{(n)}, -X_4^{(n)}, X_5^{(n)}, X_6^{(n)}, X_7^{(n)}] \, \mathrm{Ra}, \tag{32}$$

where $X_1^{(n)}, \ldots, X_7^{(n)}$ are functions of N_1, N_3, N_5, Q and q_j's defined in Appendix 1.

Substituting (20) and (21) into (24) and making use of (32), we get

$$K_{mn} = \begin{cases} [X_4^{(n)}(\phi_1^{(m)} - \phi_2^{(m)}) + X_5^{(n)}\phi_3^{(m)} + X_1^{(n)}\phi_4^{(m)} + X_2^{(n)}\phi_5^{(m)} \\ \quad + X_3^{(n)}\phi_6^{(m)} + X_6^{(n)}\phi_7^{(m)} + X_7^{(n)}\phi_8^{(m)} + \phi^{(mn)}] \, \mathrm{Ra}, \\ \qquad\qquad\qquad\qquad \text{when } m \neq n, \\ [X_4^{(m)}(\phi_1^{(m)} - \phi_2^{(m)}) + X_5^{(m)}\phi_3^{(m)} + X_1^{(m)}\phi_4^{(m)} + X_2^{(m)}\phi_5^{(m)} \\ \quad + X_3^{(m)}\phi_6^{(m)} + X_6^{(m)}\phi_7^{(m)} + X_7^{(m)}\phi_8^{(m)} + \phi_9^{(m)}] \, \mathrm{Ra}, \\ \qquad\qquad\qquad\qquad \text{when } m = n. \end{cases} \tag{33}$$

The expressions for $\phi_1^{(m)}, \ldots, \phi_9^{(m)}$ and $\phi^{(mn)}$, appearing in (33), are presented in Appendix 2.

3.1 *Determination of the critical Rayleigh number*

Equation (25) with K_{mn}'s given by (29) or (33) can be solved by including successively more rows and columns to obtain the critical value of the Rayleigh number. For numerical solution, we have considered up to the fourth order determinant. In our

computations, the following values have been chosen for the parameters:

$$N_1 = (0{\cdot}1, 0{\cdot}5, 1, 1{\cdot}5), \qquad N_3 = (2, 4, 8, 10),$$

$$N_5 = (0{\cdot}05, 0{\cdot}5, 1, 2), \qquad Q = (0, 100, 1000, 2000).$$

These values for N_1, N_3 and N_5 satisfy the nondimensional equivalent of the thermodynamic restrictions given in [8].

We find from our calculations that the roots of Ra are both positive and negative for $Q \neq 0$, whereas they are all positive for $Q = 0$. In the present analysis, we consider that the fluid is heated from below. In other words, the upper boundary is kept at a temperature lower than the lower boundary, *i.e.*, $\Delta T > 0$. Thus, from the definition of Ra in (12), we conclude that the lowest positive root of Ra with respect to the wave number a gives the critical Rayleigh number for both $Q = 0$ and $Q \neq 0$. On the other hand, the negative roots of Ra for $Q \neq 0$ corresponds to the situation when the temperature at the upper boundary is kept higher than that at the lower boundary ($\Delta T < 0$). The critical Rayleigh number for such a case is given by the maximum of all the negative roots of Ra with respect to the wave number a. Since we consider that the fluid is heated from below we are not interested in these results. However, it may be remarked here that the two branches of the Rayleigh number (one positive and one negative) with respect to a wave number separating the zones of stability have been observed more frequently for micropolar fluids [4], [6], [10] and [11], but has no classical analogue.

4. Heat generated within the fluid

If the heat is generated within the fluid, Q will be replaced by $-Q$. Writing $-Q$ for Q in (16), we have

$$(D^2 - a^2 - \Pr\sigma)\,\Theta + (W - N_5 G)\,[1 - Q(z - \tfrac{1}{2})] = 0.$$

Making use of the transformation

$$z = 1 - \zeta, \tag{34}$$

the above equation reduces to

$$(D^2 - a^2 - \Pr\sigma)\,\Theta + (W - N_5 G)\,[1 + Q(\zeta - \tfrac{1}{2})] = 0, \tag{35}$$

where $D = \mathrm{d}/\mathrm{d}\zeta$. Equation (35) is similar to (16). Since $0 \leqslant z \leqslant 1$, using the transformation (34), we get $1 \geqslant \zeta \geqslant 0$, *i.e.* the boundary conditions interchange and as such the intervals for the variation of z and ζ are the same. Thus, the boundary conditions do not change under this transformation. This means that the heat source and heat sink have the same effect in micropolar fluid. Similar phenomenon was also observed for Newtonian fluids [13].

5. Results and discussion

The effect of variations of N_1, N_3 and N_5 on the critical Rayleigh number Ra_c for various Q values has been plotted in figures 1–3, respectively. It can be seen from these figures that in both cases, namely, when the boundaries are free and when the

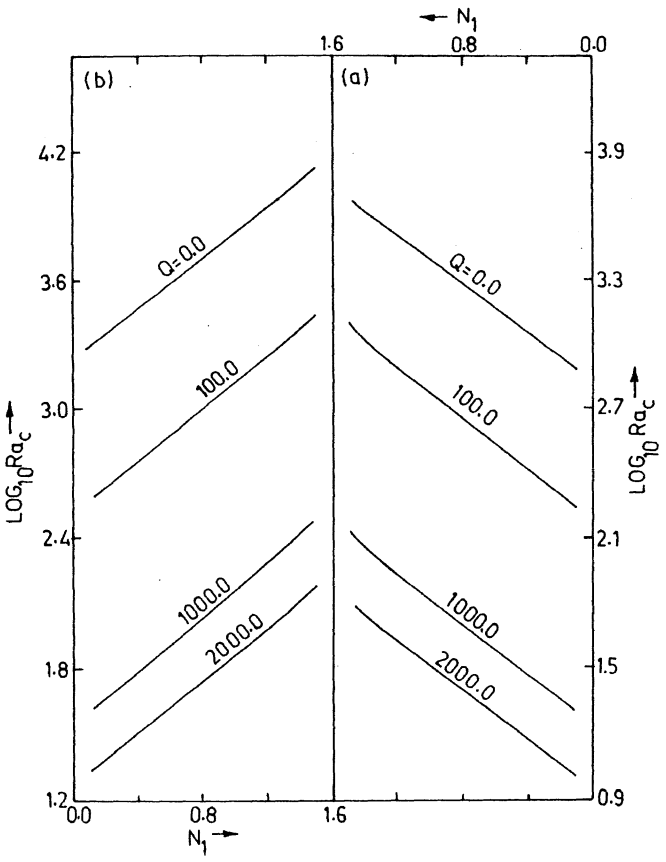

Figure 1. Critical Rayleigh number Ra_c when the boundaries are **a.** free and **b.** rigid for various values of N_1 and Q ($N_3 = 2$, $N_5 = 1$).

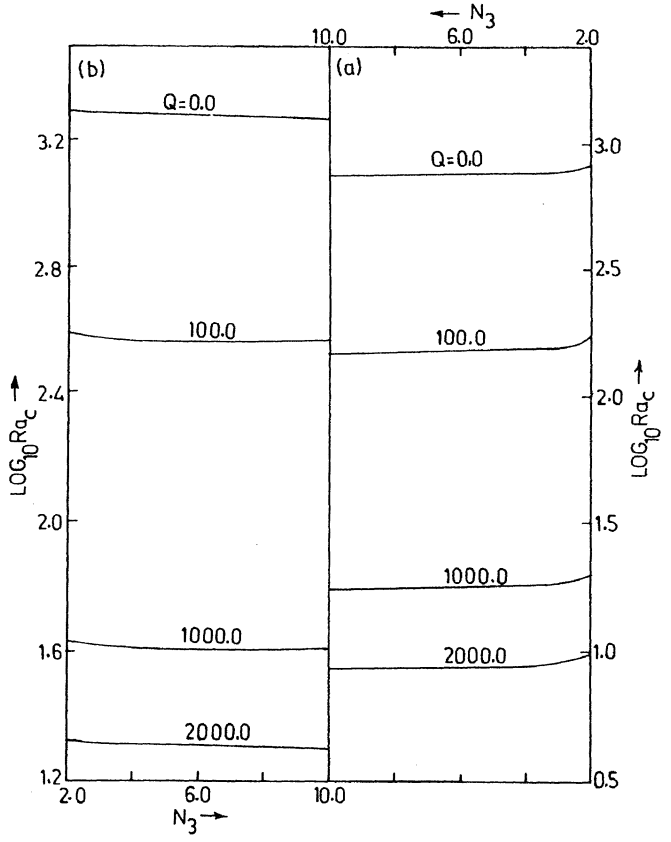

Figure 2. Critical Rayleigh number Ra_c when the boundaries are **a.** free and **b.** rigid for various values of N_3 and Q ($N_1 = 0{\cdot}1$, $N_5 = 1$).

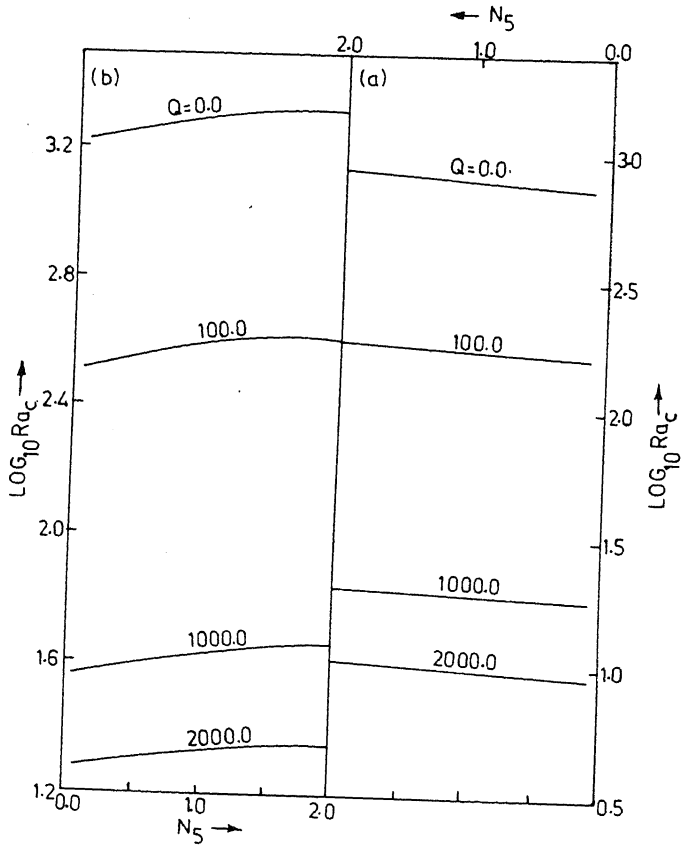

Figure 3. Critical Rayleigh number Ra_c when the boundaries are **a.** free and **b.** rigid for various values of N_5 and Q ($N_1 = 0{\cdot}1$, $N_3 = 2$).

boundaries are rigid, increase of N_1 or N_5 increases the critical Rayleigh number whereas increase of N_3 is associated with a decrease in the critical Rayleigh number. This means that increase in N_1 or N_5 delays the onset of instability while it hastens with increase in N_3.

The concentration of microelements increases with increase in N_1. Therefore, a greater part of the kinetic energy of the system is consumed in developing gyrational velocities of the fluid, and as a result, the onset of instability is delayed. When N_5 increases, the heat induced into the fluid is also increased, thus reducing the heat transfer from the bottom to the top. The decrease in heat transfer is responsible for delaying the onset of instability due to any increase in N_5. However, increase in N_3 increases the couple stress of the fluid, which causes a decrease in microrotation and hence makes the system more unstable.

Further, we observe from figures 1–3 that in both the cases as Q increases, Ra_c decreases, which implies that the layer becomes less stable. Similar behaviour of Q on Ra_c was observed by Sparrow *et al* [12], and Watson [13] for Newtonian fluids. A direct comparison with the results obtained in [13] for Newtonian fluids reveals that micropolar fluids are more sensitive to the heat source parameter.

Table 1 shows the effect of variation of N_1, N_3 and N_5 on η defined as the ratio of the critical Rayleigh number at $Q \neq 0$ to the critical Rayleigh number at $Q = 0$. It can be seen from table 1 that the values of η change significantly for any changes in N_1 or N_5 and remain the same for any changes in N_3.

In our calculation, we have observed an interesting result that the critical wave number a_c is approximately the same for various values of N_1, N_3 and N_5 for a given Q. These results agree with those obtained by Lebon and Perez-Garcia [9] when the boundaries are free and $Q = 0$. We have found that in the case of free boundaries $a_c \simeq 2{\cdot}199$ when $Q = 0$, and when $Q \geqslant 100$ (approx.), the value is 2·97 and remains approximately the same for variations of Q. For rigid boundaries also a similar result is obtained, the values of a_c being 3·087 when $Q = 0$ and 3·975 when $Q \geqslant 100$ (approx.). These results show that the horizontal dimension of the cells remains insensitive to the changes in the micropolar parameters N_1, N_3 and N_5 and also to the heat source parameter Q excepting near $Q = 0$, where the change is drastic.

In figures 4 and 5, we have plotted the vertical component of velocity profiles for free

Table 1. Effect of variation of N_1, N_3 and N_5 on η [$= (\mathrm{Ra}_c)Q \neq 0/(\mathrm{Ra}_c)Q = 0$].

	$N_1 = 0{\cdot}1$, $N_3 = 2$ or 10, $N_5 = 1$		$N_1 = 1$, $N_3 = 2$, $N_5 = 1$		$N_1 = 0{\cdot}1$, $N_3 = 2$, $N_5 = 0{\cdot}05$	
Q	η_f	η_r	η_f	η_r	η_f	η_r
100	0·2268	0·1948	0·2355	0·1988	0·2271	0·2009
1000	0·0253	0·0214	0·0263	0·0219	0·0254	0·0221
2000	0·0127	0·0108	0·0132	0·0110	0·0128	0·0111

Note: In the above table, suffixes f and r in η, respectively, correspond to the cases when the boundaries are free and when the boundaries are rigid.

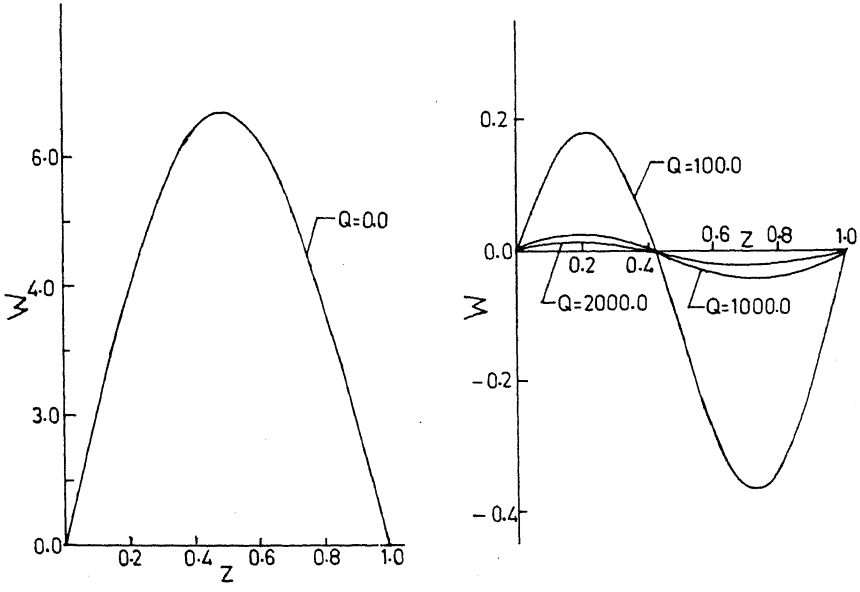

Figure 4. Vertical component of velocity profiles when the boundaries are free for $N_1 = 1{\cdot}5$, $N_3 = 2$ and $N_5 = 1$.

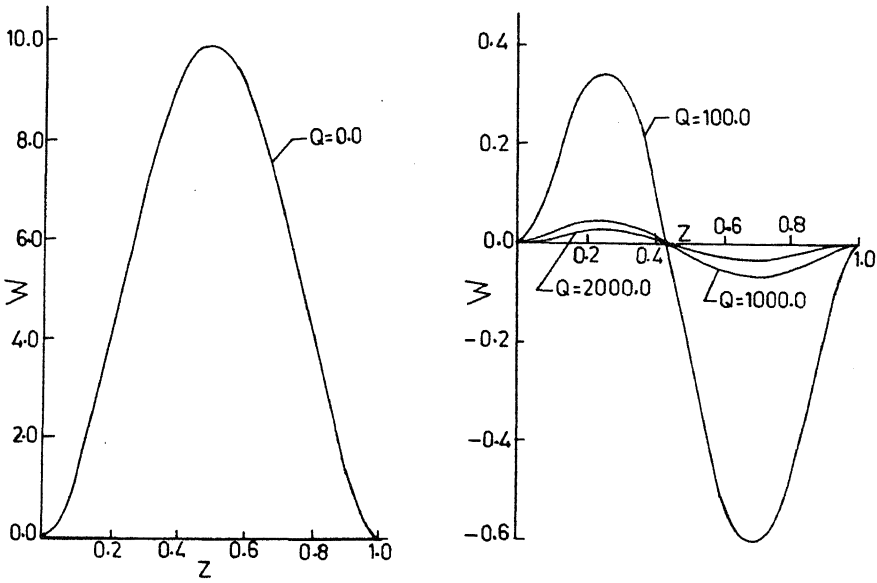

Figure 5. Vertical component of velocity profiles when the boundaries are rigid for $N_1 = 1{\cdot}5$, $N_3 = 2$ and $N_5 = 1$.

Table 2. z-component of the curl of microrotation for $N_1 = 1{\cdot}5$, $N_3 = 2$ and $N_5 = 1$.

		G							
		When the boundaries are free				When the boundaries are rigid			
z	Q	0	100	1000	2000	0	100	1000	2000
0·0		0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·1		1·3830	0·0876	0·0094	0·0047	0·8391	0·0948	0·0102	0·0051
0·2		2·6306	0·1337	0·0143	0·0072	2·7901	0·2258	0·0243	0·0122
0·3		3·6208	0·1135	0·0119	0·0059	4·8597	0·2393	0·0255	0·0127
0·4		4·2565	0·0289	0·0024	0·0011	6·3671	0·0971	0·0096	0·0047
0·5		4·4755	−0·0914	−0·0110	−0·0055	6·9139	−0·1402	−0·0165	−0·0084
0·6		4·2565	−0·2028	−0·0233	−0·0117	6·3673	−0·3564	−0·0402	−0·0203
0·7		3·6208	−0·2614	−0·0298	−0·0150	4·8597	−0·4397	−0·0492	−0·0247
0·8		2·6306	−0·2412	−0·0273	−0·0137	2·7900	−0·3434	−0·0382	−0·0192
0·9		1·3830	−0·1441	−0·0163	−0·0082	0·6388	−0·1318	−0·0146	−0·0073
1·0		0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000

boundaries and rigid boundaries respectively after taking the arbitrary constant of multiplication, in (19), to be one. From the figures, we observe that the vertical component of the velocity does not vanish anywhere between the two boundaries for $Q = 0$, but for even a small change in the value of Q, it vanishes at a point closer to the lower boundary. Moreover, the point does not change for any change in the value of Q when $Q \geqslant 100$ (approx.).

We have recorded in table 2 the values of the z-component of the curl of microrotation for different values of z. It is observed that the nature of the z-component

Table 3. Effect of variation of N_1, N_3 and N_5 on the critical Rayleigh number (Ra_c) when $\Delta T < 0$ and $Q \neq 0$.

			$-Ra_c$		
N_1	N_3	N_5	$Q = 100$	1000	2000
			When both the boundaries are free		
1·5	2	1	1657·6960	147·5143	73·2853
1.0	2	1	740·2871	66·1753	32·8843
0·1	2	1	223·2833	19·9949	10·3590
0·1	4	1	217·6056	19·4866	9·6727
0·1	10	1	214·3270	19·1931	9·5385
0·1	2	2	235·6398	21·1012	10·4868
0·1	2	0·05	212·6881	19·0462	9·4655
			When both the boundaries are rigid		
1·5	2	1	3656·8070	332·2739	163·6716
1·0	2	1	1624·3410	146·2753	72·2295
0·1	2	1	482·7146	43·6861	21·6617
0·1	4	1	468·9429	42·4259	21·1429
0·1	10	1	460·7144	41·7325	20·5935
0·1	2	2	509·2964	46·0354	22·9305
0·1	2	0·05	419·1543	37·9277	18·8887

The value of a_c is 2·977 and 3·976 in all cases when both the boundaries are free and rigid respectively.

of the curl of microrotation is the same as that shown, in figures 4 and 5, for the vertical component of the velocity profile.

So far, we have discussed the results for the case when the upper boundary is kept at a temperature lower than the lower boundary (Ra > 0). But we have observed that for $Q \neq 0$, the Rayleigh number can also be negative. As we mentioned earlier in §3, we are not interested in the case when Ra < 0. However, for the sake of completeness we have presented these results in table 3 for various values of N_1, N_3 and N_5.

Acknowledgements

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Appendix 1

$$X_1^{(n)} = Y_4^{(n)}/\lambda_{15},$$

$$X_2^{(n)} = (X_1^{(n)}\lambda_9 + Y_2^{(n)})/\lambda_{10},$$

$$X_3^{(n)} = \lambda_6(X_1^{(n)}\lambda_5 + X_2^{(n)}\lambda_3 - Y_1^{(n)}\cosh q_2)/\lambda_7,$$

$$X_4^{(n)} = -X_3^{(n)}/\lambda_6,$$

$$X_5^{(n)} = -(X_1^{(n)}q_1 + X_2^{(n)}q_2 + Y_1^{(n)}),$$

$$X_6^{(n)} = q_2(X_3^{(n)} + \lambda_6 X_4^{(n)}),$$

$$X_7^{(n)} = [q_2\lambda_6(X_4^{(n)}\cosh q_1 + X_1^{(n)}\sinh q_1) + X_5^{(n)}q_2\sinh q_2 + X_3^{(n)}q_2\cosh q_2 - X_6^{(n)}\cosh q_3]/\sinh q_3,$$

where

$$\lambda_1 = q_1 \sinh q_1 - q_2 \sinh q_2, \qquad \lambda_2 = q_2 \sinh q_2 + \cosh q_2,$$

$$\lambda_3 = \sinh q_2 - q_2 \cosh q_2, \qquad \lambda_4 = \cosh q_1 - \cosh q_2,$$

$$\lambda_5 = \sinh q_1 - q_1 \cosh q_2, \qquad \lambda_6 = (N_1 + 2)/(N_3 q_2),$$

$$\lambda_7 = \lambda_4 - \lambda_6 \sinh q_2, \qquad \lambda_8 = \lambda_6 \sinh q_1 - q_1 \sinh q_2,$$

$$\lambda_9 = \lambda_7 \lambda_8 - \lambda_6 \lambda_4 \lambda_5, \qquad \lambda_{10} = \lambda_6 \lambda_4 \lambda_3 + \lambda_7 q_2 \sinh q_2,$$

$$\lambda_{11} = q_2 \cosh q_2 + \sinh q_2, \qquad \lambda_{12} = \lambda_1 - \lambda_6 \lambda_{11},$$

$$\lambda_{13} = q_1 \cosh q_1 - q_1 \lambda_2, \qquad \lambda_{14} = q_2 \cosh q_2 - q_2 \lambda_2,$$

$$\lambda_{15} = \lambda_{10}(\lambda_{13}\lambda_7 - \lambda_{12}\lambda_5) + \lambda_9(\lambda_7 \lambda_{14} - \lambda_3 \lambda_{12}),$$

$$Y_1^{(n)} = n\pi c_n \gamma_n a^2, \qquad Y_2^{(n)} = Y_1^{(n)}(\lambda_6 \lambda_4 \cosh q_2 - \lambda_7 \sinh q_2),$$

$$Y_3^{(n)} = Y_1^{(n)}[\lambda_2 + (-1)^{n+1}], \qquad Y_4^{(n)} = \lambda_{10}(Y_3^{(n)}\lambda_7 - Y_1^{(n)}\lambda_{12}\cosh q_2) - Y_2^{(n)}(\lambda_{14}\lambda_7 - \lambda_{12}\lambda_3).$$

Appendix 2

$$\phi_1^{(m)} = (1 - \tfrac{1}{2}Q)\psi_1^{(m)} + Q\psi_7^{(m)} + N_5(N_1 + 2)[(1 - \tfrac{1}{2}Q)\psi_1^{(m)} + Q\psi_7^{(m)}]/N_3,$$

$$\phi_2^{(m)} = (1 - \tfrac{1}{2}Q)\psi_2^{(m)} + Q\psi_8^{(m)},$$

$$\phi_3^{(m)} = (1 - \tfrac{1}{2}Q)\psi_8^{(m)} + Q\psi_{13}^{(m)} + N_5 q_2[(1 - \tfrac{1}{2}Q)\psi_5^{(m)} + Q\psi_{11}^{(m)}],$$

$$\phi_4^{(m)} = (1 - \tfrac{1}{2}Q)\psi_4^{(m)} + Q\psi_{10}^{(m)} + N_5(N_1 + 2)[(1 - \tfrac{1}{2}Q)\psi_4^{(m)} + Q\psi_{10}^{(m)}]/N_3,$$

$$\phi_5^{(m)} = (1 - \tfrac{1}{2}Q)\psi_5^{(m)} + Q\psi_{11}^{(m)},$$

$$\phi_6^{(m)} = (1 - \tfrac{1}{2}Q)\psi_{11}^{(m)} + Q\psi_{14}^{(m)} + N_5 q_2[(1 - \tfrac{1}{2}Q)\psi_2^{(m)} + Q\psi_8^{(m)}],$$

$$\phi_7^{(m)} = -N_5[(1 - \tfrac{1}{2}Q)\psi_3^{(m)} + Q\psi_9^{(m)}],$$

$$\phi_8^{(m)} = -N_5[(1 - \tfrac{1}{2}Q)\psi_6^{(m)} + Q\psi_{12}^{(m)}],$$

$$\phi_9^{(m)} = \tfrac{1}{2}\gamma_m a^2[c_m - N_5 N_1(m^2\pi^2 + a^2)],$$

$$\phi^{(mn)} = \gamma_n a^2 Q\psi^{(mn)}[c_n - N_1 N_5(n^2\pi^2 + a^2)], \text{ when } m \neq n,$$

where

$$\psi_j^{(m)} = \frac{m\pi[1 + (-1)^{m+1}\cosh q_j]}{m^2\pi^2 + q_j^2}, \quad j = 1, 2, 3,$$

$$\psi_{j+3}^{(m)} = \frac{m\pi(-1)^{m+1}\sinh q_j}{m^2\pi^2 + q_j^2}, \quad j = 1, 2, 3,$$

$$\psi_{j+6}^{(m)} = \frac{m\pi(-1)^{m+1}}{m^2\pi^2 + q_j^2}\left[\cosh q_j - \frac{2q_j \sinh q_j}{m^2\pi^2 + q_j^2}\right], \quad j = 1, 2, 3,$$

$$\psi_{j+9}^{(m)} = \frac{m\pi}{m^2\pi^2 + q_j^2}\left[(-1)^{m+1}\sinh q_j - \frac{2q_j\{1 + (-1)^{m+1}\cosh q_j\}}{m^2\pi^2 + q_j^2}\right], \quad j = 1, 2, 3,$$

$$\psi_{13}^{(m)} = \frac{m\pi(-1)^{m+1}\cosh q_2}{m^2\pi^2+q_2^2} - \frac{2m\pi}{(m^2\pi^2+q_2^2)^2}\Bigg[2q_2(-1)^{m+1}\sinh q_2 + \{1+(-1)^{m+1}\cosh q_2\}\left(1-\frac{4q_2^2}{m^2\pi^2+q_2^2}\right)\Bigg],$$

$$\psi_{14}^{(m)} = \frac{m\pi(-1)^{m+1}\sinh q_2}{m^2\pi^2+q_2^2} - \frac{2m\pi(-1)^{m+1}}{(m^2\pi^2+q_2^2)^2}\Bigg[2q_2\cosh q_2 + \sinh q_2\left(1-\frac{4q_2^2}{m^2\pi^2+q_2^2}\right)\Bigg],$$

$$\psi^{(mn)} = \frac{2m\pi[(-1)^{m+n}-1]}{\pi^2(m^2-n^2)^2}, \text{ when } m \neq n.$$

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Couette flow of a stratified fluid with suction varying along a boundary

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Abstract. A viscous incompressible fluid between two plane boundaries is stratified by maintaining the planes at different temperatures. The upper plane moves with a uniform velocity. The suction/injection mechanism with constant injection velocity at the upper plane and suction velocity varying sinusoidally along the lower plane with a wave number k is introduced at the boundaries. The steady linearised equations are solved using similarity variables for the velocity components. The wave number k is shown to be effective in controlling the boundary layer thickness.

Keywords. Boundary layer control; suction; stratification; wave number; Couette flow.

1. Introduction

The flow of a viscous incompressible fluid bounded by one or two infinite planes with porous walls has gained considerable importance in view of its applications to reduce boundary layer growth. Several investigations have shown that the suction of the fluid and heat transfer serve as effective mechanisms in the boundary layer control. Stuart [3] studied the oscillatory flow past an infinite porous wall with constant suction. The free convection flow and heat transfer along a porous plane are described by Gersten and Gross [1] and Singh *et al* [2]. Venkatasiva Murthy and Ponnuraj [4] studied the Couette flow of a viscous stratified fluid between two parallel porous planes.

In this paper, the Couette flow of a viscous incompressible temperature stratified fluid is studied. The two plane boundaries are maintained at different temperatures. The lower plane is at rest and the upper plane moves with a uniform velocity. The upper plane is subjected to a constant injection and the lower plane to a suction whose magnitude varies sinusoidally with the distance along the boundary, over a mean suction. The solution of the steady linear equations is obtained using similarity variables for velocity components following Venkatasiva Murthy and Ponnuraj [4]. The wave number k appearing in the variable suction is effective in controlling the thickness of the boundary layer.

2. Governing equations and solution

The two-dimensional flow of a viscous stratified fluid between two infinite planes $z' = \pm L/2$ is considered with reference to the rectangular cartesian coordinate axes $0x'z'$. In the basic state the planes are maintained at different temperatures T_1 and T_2. The fluid is subjected to constant suction and injection at the lower and upper planes respectively. The temperature difference between the planes causes an exponential density

distribution in the fluid. If (u_e, v_e, w_e) are the velocity components, ρ_e the density and T_e the temperature in the basic state, then

$$u_e = v_e = 0, \quad w_e = -W' \text{ (constant)}.$$

$$T_e = T_0 + \left(\frac{T_1 - T_2}{2}\right) \frac{\cosh(\rho_0 C_p L W'/2K) - \exp(\rho_0 C_p W' z'/K)}{\sinh(\rho_0 C_p L W'/2K)},$$

$$\rho_e - \rho_0 = -\frac{\varepsilon \rho_0}{T_0}(T_e - T_0),$$

$$T_0 = (T_1 + T_2)/2.$$

The pressure p_e in the basic state satisfies the equation

$$\frac{\partial p_e}{\partial z} = -\rho_e g.$$

T_0 and ρ_0 are the mean values and ε is a small constant. The disturbance in the fluid, over this basic state is created when (i) the upper plane moves with velocity $\varepsilon U'$ and (ii) the normal velocities at the permeable planes $z' = \pm L/2$ are maintained at $-W'$, $-W'(1 - \varepsilon \cos k'x')$ respectively. The rate of injection at the upper plane equals the rate of suction at the lower plane over a wave length $2\pi/k'$. The equations of motion for a steady flow are

$$\rho' \bar{q}' \cdot \nabla \bar{q}' = -\nabla p' + \mu \nabla^2 \bar{q}' - \rho' g \bar{k}.$$

$$\nabla \cdot (\rho' \bar{q}') = 0.$$

$$\rho' C_p \bar{q}' \cdot \nabla T' = -K \nabla^2 T' + \theta$$

$$\rho' - \rho_e = -\frac{\varepsilon \rho_0}{T_0}(T' - T_e)$$

where θ is the viscous dissipation function and $\bar{k}$ is the unit vector in the z' direction. The flow is two-dimensional and the velocity $\bar{q}'$ has components $(u', 0, w')$. The variables ρ', p' and T' are the density, pressure and temperature respectively. The constant μ is the coefficient of viscosity. The boundary conditions are

$$u' = \varepsilon U', \quad w' = -W', \quad T' = T_1 \quad \text{at } z' = L/2$$

$$u' = 0, \quad w' = -W'(1 - \varepsilon \cos k'x'), \quad T' = T_2 \text{ at } z' = -L/2.$$

Since the stratification parameter ε is small ($\varepsilon \ll 1$) for all stable stratifications, we introduce the nondimensional variables as follows

$$u' = \varepsilon V u, \tag{1}$$

$$w' = -W' + \varepsilon V w, \tag{2}$$

$$p' = p_e + \frac{\varepsilon \rho_0 g L}{T_0}(T_1 - T_2)p, \tag{3}$$

$$T' = T_e + \varepsilon(T_1 - T_2)\phi, \tag{4}$$

where $V = [(T_1 - T_2)gL/T_0]^{1/2}$ is a constant which is of the dimension of velocity. We also write $x' = Lx$, $z' = Lz$. With these nondimensional variables and under the

Boussinesq approximation, the linearised equations of motion in the nondimensional form are

$$u_x + w_z = 0, \tag{5}$$

$$E\nabla^2 u - p_x + \beta u_z = 0, \tag{6}$$

$$E\nabla^2 w - p_z + \beta w_z = 0, \tag{7}$$

$$\frac{E}{P}\nabla^2\phi + \beta\phi_z = \frac{P\beta}{2E}\frac{\exp(-zP\beta/E)}{\sinh(P\beta/2E)}w, \tag{8}$$

where $E = \mu/\rho_0 VL$ is the inverse of Reynolds number, $P = \mu C_p/K$ is the Prandtl number, $\beta = W'/V$ is a suction parameter and $k = k'L$ is a wave number.
The continuity equation (5) suggests the form for the velocities u and w as

$$\left.\begin{aligned} w(x, z) &= -f(z)\cos kx, \\ u(x, z) &= f'(z)\frac{\sin kx}{k} + g(z), \end{aligned}\right\} \tag{9}$$

Obtaining p from (7) and substituting in (6), the form for p is

$$p(x, z) = p_0(z)\cos kx + p_1 x + p_2, \tag{10}$$

where p_1 and p_2 are constants. Equation (8) gives the form for ϕ as

$$\phi(x, z) = \phi_0(z)\cos kx \tag{11}$$

The boundary conditions are

$$\left.\begin{aligned} &f = f' = \phi_0 = 0, \quad g = U \quad \text{on } z = 1/2, \\ &f = -\beta, \quad f' = g = \phi_0 = 0 \quad \text{on } z = -1/2 \end{aligned}\right\} \tag{12}$$

where $U = U'/V$.
Equations (9)–(11) are substituted in (6) and (7) to obtain the following differential equations for f and g.

$$f^{iv} + \frac{\beta}{E}f''' - 2k^2 f'' - \frac{\beta k^2}{E}f' + k^4 f = 0, \tag{13}$$

$$g'' + \frac{\beta}{E}g' = p_1, \tag{14}$$

$$\phi_0'' + \frac{P\beta}{E}\phi_0' - k^2\phi_0 = -\frac{P^2\beta}{2E^2}\frac{\exp(-zP\beta/E)}{\sinh(PB/2E)}f \tag{15}$$

The solutions of (13)–(15) subject to boundary conditions (12) are

$$f = C_1\exp(m_1 z) + C_2\exp(m_2 z) + C_3\exp(kz) + C_4\exp(-kz), \tag{16}$$

$$g = A + B\exp(-2Nz) + \frac{p_1 E}{\beta}z, \tag{17}$$

$$\phi_0 = C_5\exp(m_1' z) + C_6\exp(m_2' z) + \psi(z), \tag{18}$$

where

$$\psi(z) = \frac{-P^2\beta}{2E^2 \sin h\,(P\beta/2E)} \sum_{i=1}^{4} \frac{C_i \exp(Q_i z)}{Q_i^2 + \frac{P\beta}{E} Q_i - k^2} \tag{19}$$

$$Q_{1,2} = m_{1,2} - P\beta/E,$$

$$Q_{3,4} = \pm k - P\beta/E,$$

$$m_{1,2} = -N \pm (N^2 + k^2)^{1/2},$$

$$m'_{1,2} = -PN \pm (P^2N^2 + k^2)^{1/2},$$

$$N = \beta/2E$$

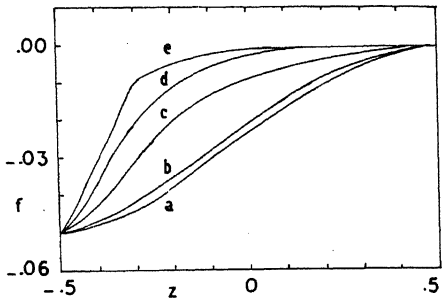
The other constants are not presented to save space.

3. Discussion

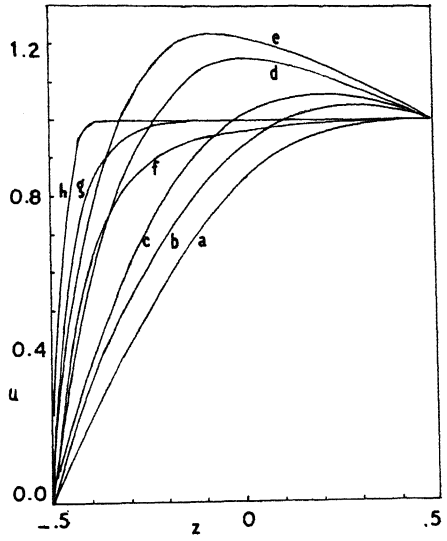
The solution given in (16) and (17) shows the existence of boundary layers of thickness orders k^{-1} and $[-N + (N^2 + k^2)^{1/2}]^{-1}$ near the upper plane and boundary layers of thickness orders k^{-1} and $[N + (N^2 + k^2)^{1/2}]^{-1}$ near the lower plane. When k is small ($k^2 \ll N^2$) the boundary layer depths at $z = 1/2$ are of orders $1/k$ and β/Ek^2 and the latter penetrates deeper than the former. Thus the effective depth of penetration is of order β/Ek^2. This layer need not be confined to small thickness near the boundary. If $k^2 \ll N^2$, the boundary layers at $z = -1/2$ have thickness orders E/β and $1/k$. If $k^2 \gg N^2$ the effective boundary layer thickness at both the planes is of order $1/k$ showing that the wave number k controls the thickness of the boundary layer. The solution (18) for temperature shows the dependence of the thermal boundary layer thickness on the relative values of the parameters P, N and k. However, large values of k correspond to thin thermal boundary layers.

For $k = 0$, the suction velocity at the lower plane is a constant $-W(1-\varepsilon)$ and the continuity equation shows the form for velocity components as $w = -f(z)$, $u = xf'(z) + g(z)$. Thus the horizontal velocity u changes linearly with x when $k = 0$. This is in contrast to the case $k \neq 0$ wherein the velocity components are given by (9). The horizontal velocity oscillates with x in the presence of k. The solution for the case $k = 0$ is developed by Venkatasiva Murthy and Ponnuraj [4]. The solution for f has parabolic dependence on z whereas for $k \neq 0$ the present solution for f consists of all exponential modes.

The behaviour of velocity and temperature and their dependence on the parameters is shown in figures 1–4. Calculations are done by prescribing $\partial p/\partial x = 0$ at $x = 0$ ($p_1 = 0$). The functions f and f' are plotted in figures 1 and 2. These figures demonstrate how the wave number k affects the thickness of the boundary layer. The profiles in figure 2 are nearly parabolic when $k = 3\pi/10, 9\pi/10$ but are rapidly distorted to steeply changing profiles near the boundary $z = -1/2$ as k increases. The curves representing f and f' in figures 1 and 2 rapidly approach zero in a layer near the lower plane and remain nearly zero throughout the channel when k is 3π or more. Figure 3 represents the curves of the horizontal velocity u as the parameter E changes. As E decreases from 0·05 to 0·01 the curves steepen and the maximum velocity attained increases beyond unity. But as E further decreases from 0·01 to 0·001 the figure reveals that the velocity tends to unity from below and the curves flatten. Thus the boundary layer thins as E



(1)



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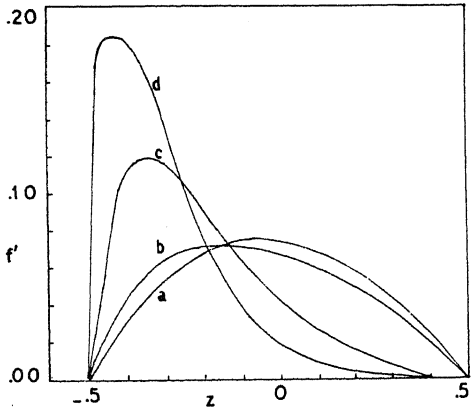

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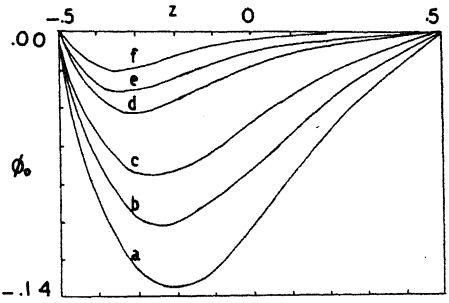

(4)

Figures 1–4. 1. The function f drawn for various k when $\beta = E = 0{\cdot}05$ **a.** $k = 5\pi/10$ **b.** $k = 9\pi/10$ **c.** $k = 2\pi$ **d.** $k = 3\pi$ **e.** $k = 4\pi$. **2.** The function f' drawn for various k when $\beta = E = 0{\cdot}05$ **a.** $k = 3\pi/10$ **b.** $k = 9\pi/10$ **c.** $k = 2\pi$ **d.** $k = 3\pi$. **3.** Velocity u drawn for various E when $\beta = 0{\cdot}05$, $k = \pi/10$ $x = 5$, **a.** $E = 0{\cdot}05$ **b.** $E = 0{\cdot}03$ **c.** $E = 0{\cdot}02$ **d.** $E = 0{\cdot}01$ **e.** $E = 0{\cdot}007$ **f.** $E = 0{\cdot}005$ **g.** $E = 0{\cdot}003$ **h.** $E = 0{\cdot}001$. **4.** Temperature ϕ_0 when $\beta = E = 0{\cdot}05$, $P = 2$ **a.** $k = 0{\cdot}05$ **b.** $k = 2$ **c.** $k = 3$ **d.** $k = 5$ **e.** $k = 6$ **f.** $k = 8$.

decreases and the velocity u is almost constant except in a very thin layer near $z = -1/2$. Figure 4 shows how the temperature ϕ_0 gets arrested as k increases.

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Numerical studies of slow viscous rotating fluid past a sphere

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Abstract. Navier-Stokes equations for steady, viscous rotating fluid, rotating about the z-axis with angular velocity ω are linearized using Stokes approximation. The linearized Navier-Stokes equations governing the axisymmetric flow can be written as three coupled partial differential equations for the stream function, vorticity and rotational velocity component. Only one parameter $R_{e\omega} = 2\omega a^2/\nu$ enters the resulting equations. Even the linearized equations are difficult to solve analytically and the method of matched asymptotic expansions is to be applied. Central differences are applied to the two-dimensional partial differential equations and are solved by the Peaceman-Rachford ADI method. The resulting algebraic equations are solved by successive over relaxation method. Streamlines are plotted for Ψ = 0·01, 0·05, and 0·25 and $R_{e\omega}$ = 0·1, 0·3, 0·5.

Keywords. Viscous rotating fluid; Navier-Stokes equation; Peaceman-Rachford method; successive over relaxation method.

1. Introduction

Navier-Stokes equations are non-linear. The problem of axi-symmetric flow past a sphere could not be solved analytically except by methods which first linearize the equations. Stokes [5] linearized the equations by neglecting inertia terms in comparison with viscous terms and solved the equations. Stokes solution is not uniformly valid throughout the flow field. Whitehead [8] tried to iterate on Stokes solution and found that condition at infinity could not be satisfied. Oseen [4] gave a solution taking into consideration inertial terms, linearly where the condition on the body is satisfied approximately. Oseen's solution has been improved by Goldstein [2] and Tomotika and Aoi [6].

The present problem consists of a study of differential equations which are a result of Stokes type of linearization on Navier-Stokes equations for steady, incompressible, viscous rotating fluid, rotating about the z-axis with a uniform angular velocity ω. We get the three coupled equations when they are expressed in stream function Ψ, vorticity ζ and rotational velocity component $\Omega = r \sin\theta V_\phi$, V_ϕ is the swirl velocity*. Hence, these equations are solved numerically by the alternating direction implicit method and the resulting algebraic equations are solved by the successive over relaxation (SOR) method [1, 7, 9].

* Even these linear equations could not be solved analytically and only method of matched asymptotic expansions is to be used.

2. Formulation of the problem

We consider a steady, slow viscous rotating fluid which is in solid body rotation with angular velocity ω about the z-axis and moving with uniform velocity U along the z-direction. A sphere of radius a is introduced into the flow and kept fixed at the origin. We use the spherical polar coordinates (r, θ, ϕ) and since the motion is axially symmetric about the z-axis, all quantities are independent of ϕ. The Navier-Stokes equations can be expressed in the form

$$\frac{1}{R_e} D^4\Psi = \frac{2\Omega}{r^2 \sin^2\theta}\left(\cos\theta \frac{\partial\Omega}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\Omega}{\partial\theta}\right) - \frac{1}{r^2\sin\theta}\frac{\partial(\Psi, D^2\Psi)}{\partial(r,\theta)} + \\ + \frac{2}{r^2\sin^2\theta} D^2\Psi\left(\cos\theta\frac{\partial\Psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\Psi}{\partial\theta}\right), \tag{1}$$

$$\frac{1}{R_e} D^2\Omega = -\frac{1}{r^2\sin\theta}\frac{\partial(\Psi,\Omega)}{\partial(r,\theta)}, \tag{2}$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right),$$

Ψ is the stream function, Reynolds number $R_e = ua/\nu$, ν is the kinematic viscosity, U is the uniform stream and ω is the angular velocity. The above equations are written in the non-dimensional form by taking

$$R_e = \frac{ua}{\nu}, \frac{\Psi'}{a^2U} = \Psi, \frac{r'}{a} = r\, v_r = \frac{u_r'}{U}, v_\theta = \frac{v_\theta'}{U}, v_\phi = \frac{v_\phi'}{U},$$

$$c = \frac{2\omega a}{U}, R_{e\omega} = \frac{2\omega a^2}{\nu}, cR_e = R_{e\omega}, \tag{3}$$

where primes denote dimensional quantities. The velocity components are given by

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \text{ and } v_\phi = \frac{\Omega}{r\sin\theta}$$

For slow translation and rotation the Navier-Stokes equations are linearized by taking v_r, v_θ, v_ϕ as $v_r, v_\theta, v_\phi + \frac{c}{2} r\sin\theta$ and neglecting the squares and products of the velocities. The linearized Navier-Stokes equations, for steady, viscous, rotating fluid, rotating about the z-axis are

$$D^4\Psi = +cR_e\left(\cos\theta\frac{\partial\Omega}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\Omega}{\partial\theta}\right) \tag{4}$$

$$D^2\Omega = -cR_e\left(\cos\theta\frac{\partial\Psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\Psi}{\partial\theta}\right) \tag{5}$$

where ζ is the vorticity.

Equations (4) and (5) are to be solved with the following boundary conditions.

$$\Omega = \Psi = \frac{\partial\Psi}{\partial r} = 0 \quad \text{on} \quad r = 1 \tag{6}$$

and

$$\Psi \sim \frac{1}{2} r^2 \sin^2\theta$$
$$\Omega \sim \frac{c}{2} r^2 \sin^2\theta \quad \text{as} \quad r \to \infty \tag{7}$$

For

$$\theta = 0°,\ 180° \quad \Psi = 0 \text{ axis of symmetry} \tag{8}$$

and for the vorticity $\zeta = \dfrac{1}{r\sin\theta} D^2\Psi$, the boundary condition

For

$$\theta = 0°,\ 180°\ \zeta = 0 \text{ axis of symmetry} \tag{9}$$

$$\zeta \to 0 \quad \text{as} \quad r \to \infty \tag{10}$$

where $c = 2\omega a/U$ and U is the uniform stream.

The condition for ζ at the surface of the sphere has to be determined from the condition of zero velocity at the surface, *i.e.*, $\dfrac{\partial\Psi}{\partial r} = 0$.

We expand the streamfunction Ψ and Ω as

$$\Psi = \Psi_0 + R_e\Psi_1 + \ldots \tag{11}$$

$$\Omega = \Omega_0 + R_e\Omega_1 + \ldots \tag{12}$$

Substituting (11) and (12) in (4) and (5) and equating like powers of R_e on both sides, we get for terms free of R_e

$$D^4\Psi_0 = 0,\ D^2\Omega_0 = 0 \tag{13}$$

with *b.c*'s

$$\Omega_0 = \Psi_0 = \frac{\partial\Psi_0}{\partial r} = 0 \quad \text{on} \quad r = 1 \tag{14}$$

$$\Psi_0 \sim \tfrac{1}{2} r^2 \sin^2\theta,\ \Omega_0 \sim \tfrac{1}{2} c r^2 \sin^2\theta \tag{15}$$

The solution to the above system of equations is

$$\Psi_0 = \frac{1}{2}\left(r^2 + \frac{3}{2}r + \frac{1}{2r}\right)\sin^2\theta \tag{16}$$

$$\Omega_0 = \frac{c}{2}\left(r^2 - \frac{1}{r}\right)\sin^2\theta \tag{17}$$

using Ψ_0 and Ω_0 from (16) and (17) the governing equations in Ψ_1 and Ω_1 are

$$D^4\Psi_1 = \frac{3c^2}{2r^2}\sin^2\theta\cos\theta \tag{18}$$

$$D^2\Omega_1 = -\frac{3c}{4}\left(1 - \frac{1}{r^2}\right)\sin^2\theta\cos\theta \tag{19}$$

with *b.c*'s

$$\Omega_1 = \Psi_1 = \frac{\partial\Psi_1}{\partial r} = 0 \text{ on } r = 1$$
$$\Psi_1 = \Omega_1 = 0 \quad \text{as} \quad r \to \infty \tag{20}$$

The solution to the above system of equations in (18) and (19) with (20) are

$$\Psi_1 = \left(A_1 + B_1 r^3 + c_1 r^5 + D_1 \frac{1}{r^2} + \frac{3c^2}{48} r^2\right) \sin^2\theta \cos\theta \tag{21}$$

$$\Omega_1 = \left[A_2 r^3 + B_2 \frac{1}{r^2} - \frac{3}{36} c(r^2 - r)\right] \sin^2\theta \cos\theta \tag{22}$$

where A_i, B_i, $i = 1, 2$ are arbitrary constants to be fixed. It is not possible to satisfy the condition at infinity on Ψ_1 and Ω_1 and hence the method of matched asymptotic expansion is to be applied for analytical solution.

3. Finite difference equations

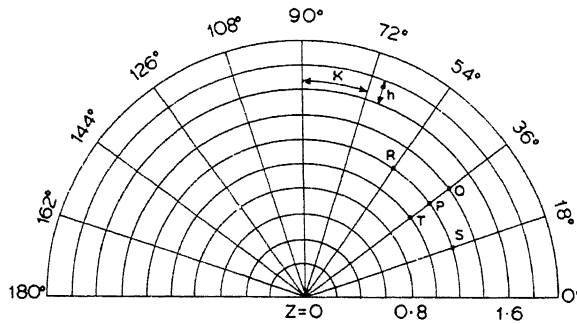
Equations (4) and (5) are written as three coupled equations

$$D^2\Psi = r \sin\theta\zeta = \zeta_1 \tag{23a}$$

$$D^2\zeta_1 = R_{e\omega}\left(\cos\theta \frac{\partial\Omega}{\partial r} - \frac{\sin\theta}{r} \frac{\partial\Omega}{\partial\theta}\right) \tag{23b}$$

$$D^2\Omega = -R_{e\omega}\left(\cos\theta \frac{\partial\Psi}{\partial r} - \frac{\sin\theta}{r} \frac{\partial\Psi}{\partial\theta}\right) \tag{23c}$$

with the boundary conditions (6)–(10). We assume that the sphere is situated on the axis of a cylindrical pipe of diameter 7.3 times the sphere diameter and the nearest lattice points to the pipe surface. We further assume the flow to be undisturbed and parallel. We choose boundary conditions at the surface of the containing pipe such that it is ensured that there is no flow through the pipe surface and the velocity gradients at the surface is zero. This prevents the establishment of a parabolic velocity distribution in the pipe which would only confuse the results. These conditions are almost equivalent to the practical case of a sphere moving with uniform velocity along the axis of a pipe. We set $r = e^2$ in (4)–(10). The lattice consists of a semi-circle with radius as $z = 2$. The lattice points are the points of intersection of the circles Z = constant and θ = constant (figure 1). For the nodal point P corresponds to $P(Z_i, \theta_j)$. $Q(Z_{i+1}, \theta_j)$ $R(Z_i, \theta_{j+1})$, $S(Z_i, \theta_{j-1})$ and $T(Z_{i-1}, \theta_j)$. The length $Z = 0$ to $Z = 2$ is divided into 9 parts with step size $h = 0.2$ and $\theta = 0$ to $\theta = \pi$ is divided into 9 parts with step length $K = 18°$. We have



The lattice

in this lattice 81 nodal points. Finite difference equations of order h^2 and K^2 are written at each point for (4)–(5) and the boundary conditions in (6)–(10). To reduce the order of the equation in (4) we write

$$D^2\Psi = r\sin\theta\zeta = \zeta_1$$

We denote any point (Z_i, θ_j) Ψ, ζ_1, and Ω by $\Psi_{i,j}$, $\zeta_{1i,j}$, and $\Omega_{i,j}$. At each nodal point (Z_i, θ_j) the finite difference equations are

$$\left(\frac{1}{h^2}-\frac{1}{2h}\right)\Psi_{i+1,j}+\left(-\frac{2}{h^2}\right)\Psi_{i,j}+\left(\frac{1}{h^2}+\frac{1}{2h}\right)\Psi_{i-1,j}$$
$$+\left(-\frac{\cot\theta_j}{2k}+\frac{1}{k^2}\right)\Psi_{i,j+1}+\left(\frac{\cot\theta_j}{2k}+\frac{1}{k^2}\right)\Psi_{i,j-1}$$
$$-\frac{2}{k^2}\Psi_{i,j}=\exp(2z_i)[\zeta_{1_{i,j}}] \tag{24}$$

$$\left(\frac{1}{h^2}-\frac{1}{2h}\right)\zeta_{1_{i+1,j}}-\frac{2}{h^2}\,\zeta_{1_{i,j}}+\left(\frac{1}{h^2}+\frac{1}{2h}\right)\zeta_{1_{i-1,j}}$$
$$+\left(-\frac{\cot\theta_j}{2k}+\frac{1}{k^2}\right)\zeta_{1_{i,j+1}}-\frac{2}{k^2}\,\zeta_{1_{i,j}}+\left(\frac{\cot\theta_j}{2k}+\frac{1}{k^2}\right)\zeta_{1_{i,j-1}}$$
$$=R_{e\omega}\exp(z_i)\left[\frac{\cos\theta_j}{2h}(\Omega_{i+1,j}-\Omega_{i-1,j})-\right.$$
$$\left.-\frac{\mathrm{Sin}\,\theta_j}{2k}(\Omega_{i,j+1}-\Omega_{i,j-1})\right] \tag{25}$$

$$\left(\frac{1}{h^2}-\frac{1}{2h}\right)\Omega_{i+1,j}-\frac{2}{h^2}\Omega_{i,j}+\left(\frac{1}{h^2}+\frac{1}{2h}\right)\Omega_{i-1,j}$$
$$+\left(-\frac{\cot\theta_j}{2k}+\frac{1}{k^2}\right)\Omega_{i,j+1}-\frac{2}{k^2}\,\Omega_{i,j}+\left(\frac{\cot\theta_j}{2k}+\frac{1}{k^2}\right)\Omega_{i,j-1}$$
$$=-R_{e\omega}\exp(z_i)\left[\frac{\cos\theta_j}{2h}(\Psi_{i+1,j}-\Psi_{i-1,j})\right.$$
$$\left.-\frac{\sin\theta}{2k}(\psi_{i,j+1}-\psi_{i,j-1})\right] \tag{26}$$

With the boundary conditions

$$\Omega_{o,j}=\Psi_{o,j}=\left(\frac{\partial\Psi}{\partial Z}\right)_{o,j}=0 \quad \text{on} \quad Z=0 \tag{27}$$

On the container pipe $Z = 2$

$$\left.\begin{aligned}\Psi_{10,j}&\to\frac{1}{2}e^4\sin^2\theta_j\\ \Omega_{10,j}&\to\frac{C}{2}e^4\sin^2\theta_j\\ \zeta_{2,j}&\to 0\end{aligned}\right\} \tag{28}$$

On the surface of the sphere, that is $Z = 0$

$$\zeta_{o,j} = \frac{8\Psi_{1,j} - \Psi_{2,j}}{2h^2 \sin\theta_j} \tag{29}$$

where ζ is the vorticity.

4. The numerical method

We have three coupled equations as governing equations. A short resume of the Peaceman-Rachford ADI method is given. Let the algebraic equations to be solved be $A \times U = b$. Then $A = H + V$ such that

$$(H + \rho I)u^{(n+1/2)} = b - (V - \rho I)u^{(n)} \tag{30}$$

(horizontal sweep)

$$(V + \rho I)u^{(n+1)} = b - (H - \rho I)u^{(n+1/2)} \tag{31}$$

(vertical sweep)

where ρ is the acceleration parameter. Computer program for this method, where lattice points are points of intersection of Z = constant circles and θ = constant lines and not as vertices of rectangles, is developed on IBM 370/155. The starting values for Ψ, Ω and ζ_1 are taken as Stokes solution for the non-rotating case. The acceleration parameter is suitably chosen for convergence. The resulting algebraic equations in (30) and (31) are solved by Block SOR method by dividing into nine blocks where each block is a tridiagonal system. One more acceleration parameter α enters for which suitable values are to be assigned.

Table 1. Solution $R_{e\omega} = 0.1$: Stream function.

Z \ θ	18	36	54	72	90	108	126	144	162
0·2	0·007513	0·016610	0·033382	0·051667	0·055608	0·049787	0·036078	0·019492	0·004485
0·4	0·018020	0·063498	0·120064	0·162639	0·181868	0·165848	0·121034	0·064871	0·021449
0·6	0·037378	0·134267	0·254904	0·348148	0·390151	0·355158	0·258641	0·137723	0·032686
0·8	0·074399	0·260152	0·494293	0·678896	0·759749	0·691831	0·504039	0·270050	0·082670
1·0	0·136752	0·474992	0·898873	1·250857	1·389663	1·265631	0·924647	0·494660	0·139559
1·2	0·237282	0·838603	1·579179	2·188837	2·438918	2·227493	1·628249	0·869340	0·243211
1·4	0·402655	1·429218	2·699664	3·748491	4·182350	3·821774	2·793807	1·491407	0·421580
1·6	0·674711	2·407434	4·556244	6·332751	7·070859	6·459272	4·718472	2·515267	0·707816
1·8	1·116113	4·021844	7·646372	10·632190	11·830177	10·782203	7·850657	4·169837	1·162612
				Solution $R_{e\omega} = 0.3$: Stream function					
0·2	0·005127	0·016010	0·040510	0·045721	0·053794	0·050219	0·041064	0·021541	0·004913
0·4	0·016073	0·059564	0·114960	0·161339	0·183536	0·171284	0·125802	0·068401	0·022687
0·6	0·035169	0·128242	0·247331	0·346037	0·394007	0·364853	0·268563	0·144311	0.034522
0·8	0·069560	0·248590	0·481479	0·671681	0·766984	0·709011	0·522123	0·281932	0·086949
1·0	0·128179	0·452205	0·872727	1·237363	1·401664	1·293521	0·958764	0·517673	0·147047
1·2	0·221084	0·795124	1·526140	2·160741	2·455565	2·281127	1·690026	0·911635	0·256899
1·4	0·373601	1·350233	2·601445	3·688698	4·206110	3·913424	2·902144	1·566171	0·446466
1·6	0·628372	2·277445	4·389960	6.227283	7.094094	6·601419	4·895259	2·639177	0·749055
1·8	1·071694	3·877290	7·433274	10·475070	11·850963	10·959363	8·081437	4·332127	1·215240

Table 2. Solution $R_{e\omega} = 0{\cdot}5$: Stream function.

Z \ θ	18	36	54	72	90	108	126	144	162
0·2	0·004763	0·012562	0·030154	0·049036	0·057128	0·054095	0·040977	0·022514	0·005379
0·4	0·015389	0·058816	0·111474	0·164456	0·190312	0·178315	0·132749	0·072484	0·023975
0·6	0·033526	0·124949	0·243138	0·349906	0·404101	0·378442	0·281886	0·152179	0·036595
0·8	0·066341	0·240405	0·472268	0·676536	0·781697	0·732835	0·545771	0·296256	0·091414
1·0	0·117318	0·434917	0·857737	1·230180	1·427270	1·341139	0·999172	0·541488	0·154329
1·2	0·203093	0·749929	1·488006	2·142708	2·496673	2·352974	1·756770	0·952878	0·270082
1·4	0·340265	1·262251	2·521836	3·632861	4·261748	4·027370	3·013041	1·637480	0·470066
1·6	0·581611	2·147014	4·237091	6·136255	7·145753	6·761685	5·077359	2·762359	0·789090
1·8	1·012390	3·710757	7·212011	10·346212	11·881587	11·142568	8·308055	4·488427	1·265388
				Solution $R_{e\omega} = 0.9$ Stream function					
0·2	0·004429	0·017636	0·035384	0·046299	0·076638	0·066583	0·048700	0·028475	0·009237
0·4	0·014469	0·056417	0·116862	0·173928	0·208110	0·200169	0·151027	0·082014	0·025299
0·6	0·030780	0·120848	0·247555	0·367155	0·437364	0·418157	0·314603	0·169739	0·046485
0·8	0·059282	0·230030	0·473380	0·702499	0·838888	0·803330	0·604134	0·331506	0·101181
1·0	0·101620	0·404009	0·843308	1·256783	1·512714	1·453560	1·093433	0·490755	0·157817
1·2	0·179410	0·679233	1·417418	2·173385	2·610125	2·521872	1·916094	1·050518	0·309163
1·4	0·292499	1·116346	2·353057	3·631200	4·404579	4.281373	3·268863	1·794810	0·515238
1·6	0·479173	1·865549	3·919613	6·019988	7·303481	7·122060	5·453883	2·997477	0·855043
1·8	0·871643	3·340050	6·785705	10·115405	12·003784	11·543239	8·755959	4·784442	1·361217

Table 3. Solution $R_{e\omega} = 0{\cdot}1$: Magnitude of vorticity vector.

Z \ θ	18	36	54	72	90	108	126	144	162
0·2	0·331060	0·576485	0·729466	0·838941	0·874893	0·829798	0·702170	0·511494	0·317620
0·4	0·362536	0·468017	0·673550	0·789684	0·802664	0·771229	0·664065	0·502597	0·321751
0·6	0·433030	0·643000	0·787369	0·896239	0·923151	0·873419	0·757993	0·599026	0·413246
0·8	0·526715	0·825834	1·011865	1·150134	1·195596	1·135353	0·978958	0·747155	0·518742
1·0	0·651923	0·972293	1·317527	1·513546	1·578109	1·499923	1·289574	0·974537	0·600020
1·2	0·837029	1·327924	1·742492	2·014646	2·105716	2·000577	1·714758	1·288962	0·786141
1·4	0·995834	1·736278	2·318265	2·701678	2·826795	2·688371	2·297253	1·700802	1·005073
1·6	1·289861	2·271659	3·097139	3·629525	3·808151	3·627491	3·096623	2·279629	1·286427
1·8	1·119712	1·868877	2·476129	2·869987	2·995095	2·844535	2·434076	1·817272	1·098197
				Solution $R_{e\omega} = 0.3$: Magnitude of vorticity vector					
0·2	0·350907	0·578986	0·774891	0·866518	0·868519	0·816825	0·684876	0·499158	0·295585
0·4	0·390679	0·532387	0·641444	0·793291	0·783821	0·752586	0·644372	0·483358	0·302326
0·6	0·473227	0·669793	0·860269	0·912309	0·930706	0·860296	0·729710	0·560385	0·387446
0·8	0·549418	0·783051	1·073483	1·172269	1·185137	1·113696	0·949054	0·718440	0·487843
1·0	0·670822	1·052164	1·286198	1·528150	1·579465	1·402951	1·260448	0·943027	0·573620
1·2	0·868577	1·386189	1·789134	2·039853	2·107829	1·985106	1·686532	1.254196	0·760739
1·4	1·001861	1·702439	2·346495	2·702442	2·823196	2·676571	2·279621	1·685721	0·988163
1·6	1·301198	2·282675	3·088648	3·624356	3·810363	3·629071	3·102627	2·248800	1·284455
1·8	1·144154	1·892707	2·518842	2·893396	2·994245	2·820875	2·396665	1·781224	1·074792

Table 4. Solution $R_{e\omega} = 0.5$: Magnitude of vorticity vector.

Z \ θ	18	36	54	72	90	108	126	144	162
0·2	0·333244	0·578938	0·765920	0·913209	0·882667	0·804312	0·658301	0·475828	0·269085
0·4	0·376575	0·543355	0·711227	0·698829	0·784756	0·732326	0·617703	0·456249	0·291081
0·6	0·494668	0·908571	0·771049	0·964345	0·980979	0·876189	0·709500	0·519214	0·350949
0·8	0·645308	0·760703	1·000002	1·228773	1·196090	1·103024	0·923592	0·686759	0·438373
1·0	0·710691	0·983099	1·468458	1·486681	1·534166	1·441365	1·225369	0·997960	0·567607
1·2	0·893732	1·447532	1·858595	2·084374	2·113117	1·969139	1·660039	1·225156	0·734981
1·4	1·023207	1·696260	2·301230	2·731561	2·825406	2·668556	2·265162	1·668300	0·962820
1·6	1·302326	2·282637	3·093200	3·613747	3·809225	3·630370	3·105437	2·286888	1·284246
1·8	1·172775	1·929996	2·545295	2·921238	2·994643	2·798613	2·361383	1·746220	1·050300
	Solution $R_{e\omega} = 0.9$ Magnitude of vorticity vector								
0·2	0·350380	0·587268	0·781956	0·880069	0·867271	0·769925	0·616678	0·432553	0·235612
0·4	0·385269	0·576710	0·715726	0·829611	0·805581	0·704387	0·560504	0·400608	0·245288
0·6	0·548900	0·782124	0·918518	0·951708	0·915562	0·809145	0·649571	0·467620	0·298160
0·8	0·691181	0·956005	1·137938	1·218063	1·189037	1·062292	0·861214	0·622973	0·384379
1·0	0·757002	1·135617	1·437396	1·590292	1·572269	1·426171	1·167169	0·845024	0·510196
1·2	0·908560	1·433507	1·853961	2·081049	2·104785	1·906876	1·607872	1·167797	0·686547
1·4	1·084341	1·794977	2·378307	2·735649	2·827949	2·648474	2·233226	1·632759	0·929629
1·6	1·312778	2·279464	3·088908	3·619126	3·810408	3·636292	3·113389	2·292551	1·280568
1·8	1·238195	2·034325	2·641294	2·966866	2·994936	2·754071	2·295293	1·682217	1·009044

Table 5. Stokes stream function.

Z \ θ	18	36	54	72	90	108	126	144	162
0·2	0·003301	0·011942	0·022620	0·031254	0·034543	0·031230	0·022581	0·011903	0·003277
0·4	0·015432	0·055828	0·105747	0·146108	0·161482	0·145993	0·105562	0·055642	0·015317
0·6	0·041156	0·148812	0·282027	0·389668	0·430672	0·389363	0·281533	0·148398	0·040851
0·8	0·087888	0·317957	0·602263	0·832130	0·919692	0·831478	0·601208	0·316902	0·087237
1·0	0·167023	0·604246	1·144541	1·581381	1·747784	1·580142	1·142536	0·602241	0·165785
1·2	0·295937	1·070626	2·027945	2·801955	3·096794	2·799759	2·024392	1·067074	0·293745
1·4	0·500992	1·812463	3·433105	4·743426	5·242560	4·739710	3·427093	1·806449	0·497280
1·6	0·822022	2·973871	5·633005	7·782968	8·601940	7·776868	5·623139	2·964004	0·815932
1·8	1·319063	4·772044	9·039041	12·488993	13·803165	12·479207	9·023210	4·756209	1·309291
	Solution: vorticity (Stokes flow)								
0·2	0·117360	0·424578	0·804221	1·111169	1·228093	1·110298	0·802813	0·423169	0·116490
0·4	0·096086	0·347615	0·658440	0·909749	1·005477	0·909036	0·657287	0·346462	0·095374
0·6	0·078669	0·284603	0·539086	0·744840	0·823217	0·744257	0·538142	0·283659	0·078086
0·8	0·064408	0·233014	0·441366	0·601824	0·673993	0·609346	0·440593	0·232240	0·063931
1·0	0·052733	0·190775	0·361360	0·499281	0·551819	0·490890	0·360727	0·190142	0·052342
1·2	0·043174	0·156194	0·295857	0·408777	0·451791	0·408457	0·295339	0·155674	0·042854
1·4	0·035348	0·127881	0·242227	0·334678	0·369895	0·334416	0·241803	0·127456	0·035086
1·6	0·028941	0·104700	0·198319	0·274011	0·302845	0·273797	0·197971	0·104356	0·028726
1·8	0·023695	0·085721	0·162370	0·224342	0·247948	0·224166	0·162085	0·085437	0·023519

5. Results

For $R_{e\omega} = 0{\cdot}1, 0{\cdot}3, 0{\cdot}5, 0{\cdot}9$ and $c = 1$ the stream function Ψ is given in the tables 1 and 2. The vorticity components ξ, η, ζ are given by

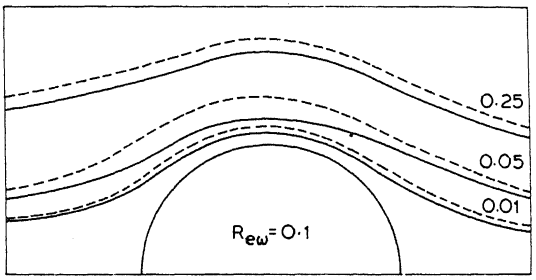
$$\xi = \frac{1}{r^2 \sin\theta}\frac{\partial\Omega}{\partial\theta}, \eta = \frac{1}{r\sin\theta}\frac{\partial\Omega}{\partial r}, \zeta = \frac{1}{r^2\sin\theta} D^2\Psi$$

Hence the magnitude of the vorticity vector is equal to

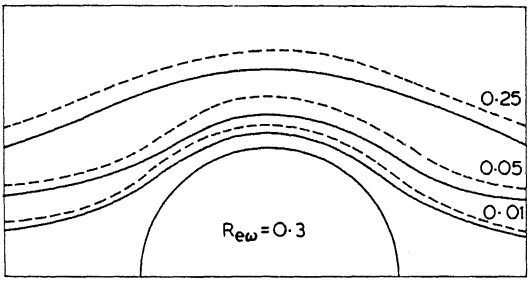
$$(\xi^2 + \eta^2 + \zeta^2)^{1/2}.$$

For the same values of $R_{e\omega}$, the magnitude of the vorticity vector is given in tables 3 and 4. In table 5 the Stokes stream function and vorticity for the non-rotating case are given so as to compare with the rotating case.

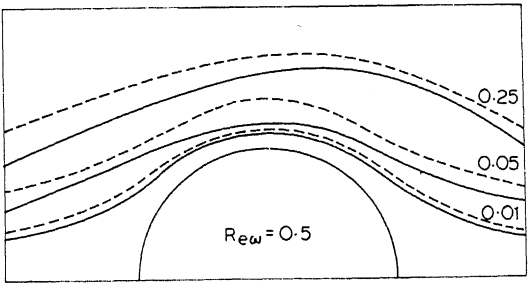
The effect of rotation is to increase the values of magnitude of vorticity. Stream lines are plotted in figures 1–3 for $R_{e\omega} = 0{\cdot}1, 0{\cdot}3, 0{\cdot}5, 0{\cdot}9$ for $\Psi = 0{\cdot}01, 0{\cdot}05, 0{\cdot}25$. The dotted lines in each figure gives the Stokes stream lines for the same values of Ψ.



(1)



(2)



(3)

Figures 1–3. Streamlines.

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Cohomology of discontinuous groups

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Abstract. Prasad (1979) proved that the set of all equivalence classes of representations ρ of a Fuchsian group Γ whose restrictions to the cyclic subgroups $\Gamma_i = (c_i)$ corresponding to the parabolic and elliptic elements of Γ occurring in the structure of Γ, are given, is a complex analytic manifold. In the process the author has proved that $H^1(X, \mathscr{A}) \approx P^1(\Gamma, \rho)$ and $\chi(X, \mathscr{A}) = (2-2g)d - \sum_{i=1}^{n} e_i$ with suitable notation.

This paper gives the corresponding results to the two above mentioned results, when in place of Γ we consider any discontinuous group of Poincare isometries Δ, and when similar assumptions are made.

Keywords. Fuchsian group; representation; cohomology spaces; discontinuous group; Riemann surface; Klein surface; sheaf; spectral sequence; Euler Poincare characteristic.

Resume of the results

et Δ be a discontinuous group of analytic and antianalytic automorphisms of the pper half plane H. The analytic part of Δ is a Fuchsian group Γ, which acts not only on but also on the set H^+ consisting of the points of H together with the parabolic fixed ints of Γ.

The author shows that Δ also acts on H^+ in a natural way and assumes that $\Gamma \neq \Delta$ id that the quotient $Y = H^+/\Delta$ which is a non-orientable Klein surface (see [2]), is mpact and has an empty boundary.

Let $\tau: \Delta \to \mathrm{Aut}_C W$ be a finite dimensional representation of Δ in a complex vector ace W. Let $\mathscr{B}$ be the Δ-invariant direct image of the constant sheaf on H^+ defined by e vector space W. Then the results presented here extend to parts (1) and (2) of the main eorem in [12], from Γ to Δ. Here X is replaced by Y, $\mathscr{A}$ by $\mathscr{B}$ and $P^1(\Gamma, \tau|\Gamma)$ by a bspace $Q^1(\Delta, \tau)$ of $H^1(\Delta, \tau)$ obtained in a similar manner. Also in part (1) the Euler aracteristic $(2-2g)$ of X is replaced by the Euler characteristic $(2-h)$ of Y(h is the nus of Y) and in part (2) the constants e_i are replaced by constants f_j defined in a nilar manner.

Precisely,

) $H^1(Y, \mathscr{B}) \approx Q^1(\Delta, \tau)$

d

) $\chi(Y, \mathscr{B}) = (2-h)d - \sum_{j=1}^{m} f_j$ where $H^1(Y, \mathscr{B})$ is the first cohomology space of the

sheaf $\mathscr{B}$ and $\chi(Y, \mathscr{B})$ represents the Euler Poincare characteristic of $\mathscr{B}$. The dimension of W is denoted by d.

2. Notation and some preliminary observations

Let H denote the upper half plane and Δ be a discontinuous group (see [5] or [8]) of analytic and anti-analytic automorphisms of H where the subset of anti-analytic automorphisms is non-empty.

We note that every analytic automorphism of H is of the type

$$z \to \frac{az+b}{cz+d}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix of real numbers of determinant 1. Every anti-analytic automorphism of H is of the type

$$z \to \frac{a\bar{z}+b}{c\bar{z}+d}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix of real numbers of determinant -1.

Because of the discontinuity of Δ and the fact that the set of anti-analytic automorphisms in Δ is non-empty, it follows that the quotient $V = H/\Delta$ is a non-orientable Klein surface (Theorem 1.8.4 of [2]).

Let Γ denote the subgroup of Δ consisting of precisely the analytic automorphisms of H, Then Γ is a normal subgroup of index two in Δ and is in fact a Fuchsian group (see [8], [9]). Thus $U = H/\Gamma$ is a Riemann surface. If we adjoin the set P of parabolic fixed points of Γ to H, and call the union H^+, then Γ acts on H^+ and the quotient $X = H^+/\Gamma$ is also a Riemann surface containing U as an open subset.

We observe now that Δ also acts on P and hence also on H^+. To see this, let $p \in P$ and $\delta \in \Delta$. Then $\delta(p) \in R \cup \{-\infty, \infty\}$ and if $\gamma \in \Gamma$ leaves the point p fixed and is parabolic, then $\delta\gamma\delta^{-1}$ has for its unique fixed point $\delta(p)$, and hence $\delta(p) \in P$. We denote the quotient H^+/Δ by Y. Obviously, V is an open subset of Y.

It is equally clear that if $H^{(-)} = H - E$, where E is the set of elliptic fixed points of Γ, then Γ and Δ act on $H^{(-)}$. The set $U_1 = H^{(-)}/\Gamma$ is open in X and $V_1 = H^{(-)}/\Delta$ is an open subset of Y. Needless to say that V_1 is a non-orientable Klein surface. We have a commutative diagram of canonical maps

$$\text{(I)} \qquad \begin{array}{ccccc} H^{(-)} & \to H^{(-)}/\Gamma & = U_1 & \to H^{(-)}/\Delta & = V_1 \\ \cap & & \cap & & \cap \\ H & \to H/\Gamma & = U & \to H/\Delta & = V \\ \cap & & \cap & & \cap \\ H^+ & \xrightarrow{f} H^+/\Gamma & = X & \xrightarrow{g} H^+/\Delta & = Y \end{array}$$

We now consider the (algebraic) condition on Δ; (A) For every $\delta \in \Delta - \Gamma$, $\delta^2 \neq 1$ and the (topological) condition on V:
(A') The Klein surface V has no boundary.
We state and prove the following Lemma:

Lemma: (a) Conditions (A) and (A') are equivalent. (b) If condition (A) holds, each fiber of g consists of exactly 2 elements. (c) Y is a non-orientable Klein surface and g is a two-sheeted unramified double cover (in the sense of [2]). (d) Conditions (A) and (A') are equivalent to the condition:

(A'') Y has no boundary.

Proof: Let $\delta^2 = 1$ for some $\delta \in \Delta - \Gamma$. If we can show that there is at least one fixed point of δ in H, then the proof of theorem 1.8.4 of [2] implies that V has a non-empty boundary.

We shall show that δ has a fixed point $z = x + iy \in H$. Let δ be given by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $ad - bc = -1$. Then δ has a fixed point $z = x + iy \in H$

$$\Leftrightarrow \frac{a\bar{z} + b}{c\bar{z} + d} = z \in H$$

$$\Leftrightarrow d + a = 0 \tag{1}$$

and $\exists\ x$ and $y > 0$ such that

$$c(x^2 + y^2) + (d - a)x - b = 0 \tag{2}$$

Now $\delta^2 = 1$ implies that $a^2 + bc = \pm 1 = bc + d^2$
and $(a + d)b = 0 = (a + d)c$

so that

$$\begin{aligned} 2 = |\mathrm{tr}(I)| &= |\mathrm{tr}(\delta^2)| \\ &= |(a^2 + bc) + (bc + d^2)| \\ &= |(a + d)^2 + 2(bc - ad)| \\ &= |(a + d)^2 + 2| \end{aligned}$$

so that $a + d = 0$ proving (1).

To see (2), we see that if $c = 0$, then (2) reduces to

$$(d - a)x - b = 0$$

and $x = b/(d - a)$ (trivially $d - a \neq 0$)

Thus all the points on the line $x = b/(d - a)$ are fixed points of δ.
If $c \neq 0$, (2) reduces to

$$\left(x + \frac{d - a}{2c}\right)^2 + y^2 = \frac{b}{c} + \left(\frac{d - a}{2c}\right)^2$$

and (2) will be proved if we can show that

$$\frac{b}{c} + \left(\frac{d - a}{2c}\right)^2 > 0$$

This latter condition is equivalent to

$$4bc+(d-a)^2>0$$

or $4(ad+1)+(d-a)^2>0$

or $4+(d+a)^2>0$

or $4>0.$

which is true.

Thus all the points on the semicircle

$$\left(x+\frac{d-a}{2c}\right)^2+y^2=\frac{b}{c}+\left(\frac{d-a}{2c}\right)^2, \quad y>0$$

are fixed by δ and (2) is proved.

Thus V has non-empty boundary.

Conversely, let $\delta^2\neq 1\ \forall\delta\in\Delta-\Gamma$. We shall show that V has no boundary.

By examining the proof of theorem 1.8.4 of [2], it is clear that the only case when $V=H/\Delta$ can have boundary is when $x\notin\partial H$ (which is always true) and $\Gamma_x\neq\Delta_x$ for some $x\in H$. That is, there should exist a $\delta\in\Delta-\Gamma$ such that $\delta(x)=x$ for some $x\in H$. But then $\delta^2(x)=x$ which means that $\delta^2=1$ or δ^2 is elliptic. Above we had seen that

$$|\mathrm{tr}\,(\delta^2)|=(a+d)^2+2$$

$$\geqslant 2$$

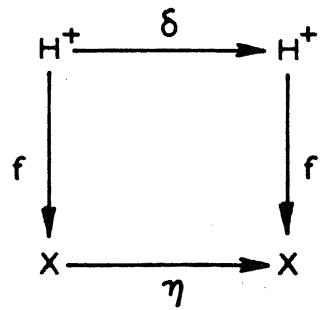
which means that δ^2 (which is in Γ) cannot be elliptic. On the other hand we are given that $\delta^2\neq 1$. Hence V can have no boundary and (a) is proved.

(b) Let us assume condition (A). We have only to show that $\Gamma_x=\Delta_x\ \forall x\in H^+$. To see this, if $\delta\in\Delta_x-\Gamma_x$ and $\delta^2\neq 1$, then δ^2 is an elliptic or parabolic transformation. But in the proof of (a), we have seen that if $\delta\in\Delta-\Gamma$, and $\delta^2\neq 1$, then δ^2 is always hyperbolic. This proves (b).

(c) We shall define an anti-analytic involution η on X such that $Y=X/\eta$. Since X is a Riemann surface, by (b) and Proposition 1.9.1 of [2], it follows that Y is a Klein surface and g is an unramified double cover. That Y is nonorientable follows because $V\subset Y$ is nonorientable.

Let $\delta\in\Delta-\Gamma$. Define $\eta(\bar{x})=\overline{\delta(x)}$ for $x\in H^+$, where $\bar{x}$ denotes the image of x in $H^+/\Gamma=X$. It is easily seen that $\eta(\bar{x})$ is independent of the specific element x chosen to represent $\bar{x}$ and is independent of δ.

As $\eta^2(\bar{x})=\overline{\delta^2(x)}=\bar{x}$, $\eta^2=1$, *i.e.* η is an involution. As the diagram (I)



is commutative, by passing into the quotient, η is continuous. As $\delta(H) \subset H$, we have $\eta(U) \subset U$. Since f is an analytic map and $\delta|_H$ is anti analytic, $f \circ \delta|_H$ is anti-analytic. But $\eta \circ f = f \circ \delta$ and $U = H/\Gamma$. Thus by the universal nature of H/Γ, $\eta|_U$ is anti-analytic. As $X - U$ is a discrete subset of X and η is continuous on X, therefore η is anti-analytic on X.

It is obvious that $X/\eta = Y$.

(d) This is clear since $Y - V$ is a discrete subset of Y.

Q.E.D.

From now on *we assume* (A) *and that* (B) *Y is compact.*

Thus Y is a compact nonorientable Klein surface without boundary. As g is a two-sheeted covering, (B) implies that X is also compact (*vice-versa* is also clear). It follows that Γ is a Fuchsian group of the first kind—*i.e.* a horocyclic group. Also a normal polygon for Γ cannot have free sides (see [8]). The sets $X - U$ and $Y - V$ are finite. Let g be the genus of X and h the genus of Y. If $\chi(X)$ and $\chi(Y)$ are the Euler characteristics of X and Y respectively, we have

$$\chi(X) = 2 - 2g$$

and $$\chi(Y) = 2 - h.$$

It can be seen by considering a triangulation of Y and lifting it to X that

$$\chi(X) = 2\chi(Y)$$

(see theorem 3, p. 144 of [17] or Proposition 1.6.9, p. 43 of [2]). In other words

$$2 - 2g = 2(2 - h) \tag{3}$$

or $$h = g + 1.$$

It is well known from the theory of Fuchsian groups (see [9]) in our case (X: compact) that Γ can be determined by generators

$$a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_n,$$

for suitable n and relations

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots\ldots a_g b_g a_g^{-1} b_g^{-1} c_1 \ldots c_n = 1,$$

and $c_i^{s_i} = 1$ for suitable integers $s_i \geqslant 0$, $i = 1, \ldots, n$.

Those c_i for which $s_i = 0$ give rise to the various equivalence classes in Γ of parabolic transformations and those c_i with $s_i > 0$ yield the equivalence classes of elliptic transformations. If $\delta \in \Delta - \Gamma$ then $\gamma \in \Gamma$ is elliptic or parabolic iff $\delta\gamma\delta^{-1}$ is elliptic or parabolic respectively. Moreover $\delta\gamma\delta^{-1}$ is not Γ-equivalent to γ because δ does not fix any point of H^+. In other words, the equivalence classes in Γ determined by $\delta\gamma\delta^{-1}$ and γ in Γ are distinct *i.e.* there are an even number of elliptic equivalence classes and an even number of parabolic equivalence classes. In particular n is even and there is a permutation ν of $\{1, 2, \ldots, n\}$ of order 2 such that $c_{\nu(i)}$ is Δ-conjugate to c_i and $s_{\nu(i)} = s_i$, $i = 1, \ldots, n$.

Let $n = 2m$. Choose $i_1, i_2, \ldots, i_m$ such that $1 \leqslant i_k \leqslant n$ for every k and

$$\{i_1, i_2, \ldots, i_m\} \cup \{\nu(i_1), \ldots, \nu(i_m)\} = \{1, 2, \ldots, n\}.$$

Let x_i be the fixed point of c_i in H^+ and $\Gamma_i = (c_i)$ be the cyclic subgroup of Γ generated

c_i. Moreover $\Gamma_i = \Gamma_{x_i}$. The set of fixed point images in X is $\{\bar{x}_1, \ldots, \bar{x}_n\}$ and its ge in Y is in turn $\{y_{i_1}, \ldots, y_{i_m}\}$ where $y_{i_k} = \overline{\bar{x}_{i_k}} = \bar{\bar{x}}_{\nu(i_k)}$, a double line above a point x H^+ denoting its image in Y. Let $\Delta_k = \Gamma_{i_k}$, $1 \leqslant k \leqslant m$ and $\tau_k = \tau|\Delta_k$.

Statement of the main theorem

now τ be a representation of Δ in a complex vector space W of (finite) dimension d. the restriction of τ to Γ be denoted by ρ. Then we have the cohomology vector ces $H^p(\Delta, \tau)$ and $H^p(\Gamma, \rho)$ for $p \geqslant 0$, and $H^p(\Delta_k, \tau_k)$ and $H^p(\Gamma_{i_k}, \rho_{i_k})$. There is a mmutative diagram where all the maps are canonical (restriction) maps:

$$\begin{array}{ccc} H^p(\Delta, \tau) & \xrightarrow{\beta_k^p} & H^p(\Delta_k, \tau_k) \\ \Big\downarrow{\scriptstyle r_p} & & \Big\downarrow \\ H^p(\Gamma, \rho) & \xrightarrow{\alpha_k^p} & H^p(\Gamma_{i_k}, \rho_{i_k}). \end{array}$$

course, it is needless to say that there exist natural maps

$$\alpha_i^p : H^p(\Gamma, \rho) \to H^p(\Gamma_i, \rho_i).$$

n [12], we had denoted by $P^1(\Gamma, \rho)$ the space $\bigcap_{i=1}^{n} \ker \alpha_i$, P denoting that the omology classes come from parabolic cocycles (see [19]). Right now we consider analogous space

$$Q^1(\Delta, \tau) = \bigcap_{i=1}^{m} \ker \beta_k.$$

have a canoncial linear map

$$r_1 : Q^1(\Delta, \tau) \to P^1(\Gamma, \rho).$$

et $\mathscr{W}$ be the constant sheaf of vector spaces defined by W on H^+. Then $\mathscr{W}$ is a Δ-af ([5], §5.1) and gives rise to sheaves of vector spaces $\mathscr{A} = f_*^{\Gamma}(\mathscr{W})$ on X and $\mathscr{B}$ $= {}^{\Delta}_{*}(\mathscr{W})$ on Y as in [5]. If η is also used to denote the cyclic group of two elements erated by itself (considered as a group of transformations acting on X) we have $\mathscr{B}$ $= {}^{\eta}_{*}(\mathscr{A})$. We have also the cohomology vector spaces $H^p(X, \mathscr{A})$ and $H^p(Y, \mathscr{B})$ of X coefficients in $\mathscr{A}$ and Y with coefficients in $\mathscr{B}$. Let $\chi(X, \mathscr{A})$ and $\chi(Y, \mathscr{B})$ denote the er characteristics of $\mathscr{A}$ and $\mathscr{B}$ respectively. Finally, let

f_k rank of the endomorphism $\{1_W - \tau(c_{i_k})\}$ of W,

$$k = 1, \ldots, m.$$

n we have the theorem:

heorem: Under the assumptions (A) and (B),

$$H^1(Y, \mathscr{B}) \approx Q^1(\Delta, \tau)$$

$$\chi(Y, \mathscr{B}) = (2-h)d - \sum_{k=1}^{m} f_k.$$

(3) η induces an action on $H^p(X, \mathscr{A})$ and the fixed point subspace $[H^p(X, \mathscr{A})]^\eta$ is canonically isomorphic to $H^p(Y, \mathscr{B})$.

Corollary: $\chi(X, \mathscr{A}) = 2\chi(Y, \mathscr{B})$.

4. Proof of the theorem and corollary

We always keep in mind the diagram (I). Our proof of parts (1) and (2) of the theorem is modelled on the proof of parts (1) and (2) of the theorem of [12], and in particular we have analogues of Lemmas 1, 2 and 3 for the proof of part (1). We give no detailed explanations in the proof as the arguments are exactly similar and we replace Γ by Δ, ρ by τ, $\mathscr{A}$ by $\mathscr{B}$, f by $g \circ f = h$ (say) and Γ_i by Δ_{i_k} etc.

Proof of Part (1)

Let U' be an open subset of H^+. We say that U' satisfies the *condition* (D) *w.r.t.* Δ if

(i) in case U' does not contain any fixed point, $U' \cap \delta U' = \phi$ for all $\delta \in \Delta$, $\delta \neq 1$,
(ii) in case x is a fixed point in U', then $U' \cap \gamma U' = \phi$ for all $\delta \notin \Delta_x$.

We say that U' is saturated for the action of Δ_x if $\delta U' = U' \ \forall \delta \in \Delta_x$.

Lemma 1 (a) $\mathscr{B}|_V = h_*^\Delta(\mathscr{W}|_H)$ and $\mathscr{B}|_{V_1} = h_*^\Delta(\mathscr{W}|_{H^{(-)}})$

(b) Let U' be an open subset of H^+ satisfying (D) w.r.t. Δ. Let $U = h(U')$. Then if U' does not contain fixed points, U is homeomorphic to U' and $\mathscr{B}|_U$ is isomorphic to the constant sheaf and if U' contains a fixed point x and is saturated for the action of Δ_x, then U is homeomorphic to $U'|\Delta_x$ and $\mathscr{B}|_U \approx h_*^{\Delta_x}(\mathscr{W}|_{U'})$.

(c) $\mathscr{B}|_{V_1}$ is a local system.

(d) If $i: V \to Y$ and $j: V_1 \to Y$ are the canonical inclusions, then $\mathscr{B} = i_*(\mathscr{B}|_V) = j_*(\beta|_{V_1})$ where i_* and j_* denote the usual direct image (see Grothendieck [5]).

We shall say that an open set U in X is 'sufficiently small' if there is a U' open in H^+ satisfying (D) w.r.t. Δ such that $U = h(U')$.

We have

Lemma 2 (local vanishing) Let U_0 be a 'sufficiently small' open set in Y which is homeomorphic to a disc. Then

$$H^q(U_0, \mathscr{B}) = 0, \ q \geqslant 1.$$

Proof: Similar to Lemma 2 of [12].

Lemma 3 $H^1(Y, \mathscr{B}) \approx Q^1(\Delta, \tau)$.

Proof: Let $J = \{k | 1 \leqslant k \leqslant m \text{ and } c_{i_k} \text{ is parabolic}\}$. Fix $k \in J$. Let x_k be the unique fixed point of c_{i_k}. Let U_k' be the region bounded by a suitable horocycle at the point x_k together with the point x_k, chosen in such a way that U_k' satisfies condition (D) w.r.t. Δ. Obviously U_k' is saturated for the action of Δ_k. Let $U_k = h(U_k')$. By Lemma 2,

$$H^1(U_k, \mathscr{B}) = 0, \quad k \in J.$$

It can be shown that $H^1(V, \mathscr{B}) \approx H^1(\Delta, \tau)$ and $H^1(V_k, \mathscr{B}) \approx H^1(\Delta_k, \tau_k)$ where $V_k = U_k - \{\bar{x}_k\}$. The diagram

$$\begin{array}{ccc} H^1(V, \mathscr{B}) & \approx & H^1(\Delta, \tau) \\ \downarrow s_k & & \downarrow \beta_k' \\ H^1(V_k, \mathscr{B}) & \approx & H^1(\Delta_k, \tau_k) \end{array}$$

is commutative where s_k is the canonical map. Naturally

$$Q^1(\Delta, \tau) \approx \bigcap_{k=1}^{m} \ker s_k \tag{4}$$

Now, since $\mathscr{B} = i_*(\mathscr{B}|_V)$, we can apply theorem 3.7.3 of [5] and case 2 of [3], Chapter XV (§6) and obtain an exact sequence

$$0 \to H^1(Y, \mathscr{B}) \to H^1(V, \mathscr{B}) \xrightarrow{h} H^0(Y, R^1 i_*(\mathscr{B}|_V)) \downarrow H^2(Y, \mathscr{B}) \downarrow H^2(V, \mathscr{B})$$

There is a natural isomorphism

$$\prod_{k \in \mathbf{j}} H^1(V_k, \mathscr{B}) \approx H^0(Y, R^1 i_*(\mathscr{B}|_V))$$

and a factorization of h

$$\begin{array}{ccc} H^1(V, \mathscr{B}) & \xrightarrow{h} & H^0(Y, R^1 i_*(\mathscr{B}|_V)) \\ & \searrow & \int\int \\ \prod_{k \in J} S_k & & \prod_{k \in J} H^1(V_k, \mathscr{B}) \end{array}$$

Thus $\ker h = \bigcap_{k \in J} \ker s_k$ and in view of (4) it follows that $H^1(Y, \mathscr{B}) \approx Q^1(\Delta, \tau)$. This proves Lemma 3 and also part (1) of the theorem.

Proof of Part (2)

Choose an open covering of Y, say $v = \{V_k\}_{k=1}^{M}$, which is the star covering of a triangulation T of Y satisfying the conditions (a), (b), (c), (d) of §4, of [12] where we replace N by M, n by m, and the set $\{x_1, \ldots, x_n\}$ by $\{y_1, \ldots, y_m\}$, where $y_k = g \circ f(x_k)$, x_k being the unique fixed point of c_{i_k}, $1 \leqslant k \leqslant m$. We also replace U_i by V_k. Then v is $\mathscr{B}$-acyclic (Lemma 2) and also $\mathscr{C}$-acyclic where, by $\mathscr{C}$ we denote the constant sheaf of complex numbers on Y.

We have an alternating cochain resolution of $\mathscr{B}$, say

$$0 \to C^0(v, \mathscr{B}) \to C^1(v, \mathscr{B}) \to \ldots .$$

where $C^p(v, \mathscr{B}) = 0$ for $p > M$. We have the Euler-Poincare characteristic

$$\chi(Y, \mathscr{B}) = \sum_{p=0}^{M} (-1)^p \dim H^p(Y, \mathscr{B}).$$

$$= \sum_{p=0}^{M} (-1)^p \dim H^p(v, \mathscr{B})$$

$$= \sum_{p=0}^{M} (-1)^p \dim C^p(v, \mathscr{B})$$

$$= \left[\sum_{p=0}^{M} (-1)^p \dim C^p(v, \mathscr{C}) \right] \cdot \dim W$$

$$- \sum_{k=1}^{m} (\dim W - \dim W^{\Delta k})$$

$$= \chi(Y, \mathscr{C}) \cdot d - \sum_{k=1}^{m} f_k$$

$$= (2-h) \cdot d - \sum_{k=1}^{m} f_k$$

by Euler-Poincare formula, which proves part (2) of the theorem.

Proof of Part (3):

That η induces an action of the group $\{1, \eta\}$ on X and hence on $H^p(X, \mathscr{A})$ is obvious. η has no fixed points, by theorem 5.3.1 and corollary to proposition 5.2.3 of [5], we ha canonical isomorphisms

$$H^p(Y, \mathscr{B}) \approx [H^p(X, \mathscr{A})]^{\eta}.$$

In fact these isomorphisms follow in several ways from the results of §§ 5.2, 5.3 of [5

Proof of corollary: It is obvious from Part (2) and the facts that $n = 2m$ and $2 - 2$ $= 2(2-h)$.

Q.E.

Remarks: Let σ be the isomorphism from $P^1(\Gamma, \rho)$ to $H^1(X, \mathscr{A})$ deduced in t theorem of [12]. Then one feels that there exists an involution of $P^1(\Gamma, \rho)$ (mo generally, perhaps, of $H^1(\Gamma, \rho)$) say η_1 such that $[P^1(\Gamma, \rho)]^{\eta_1} = Q^1(\Gamma, \rho)$ and th $\sigma[Q^1(\Delta, \tau)] = H^1(Y, \mathscr{B})$, and that the diagram

$$\begin{array}{ccc} P^1(\Gamma, \rho) & \xrightarrow{\sigma} & H^1(X, \mathscr{A}) \\ \downarrow \eta_1 & & \downarrow \eta \\ P^1(\Gamma, \rho) & \to & H^1(X, \mathscr{A}) \end{array}$$

is commutative.

If there exists such an involution, it will follow that $\sigma|_{Q^1(\Delta, \tau)}$ is an isomorphism $Q^1(\Delta, \tau)$ onto $H^1(Y, \mathscr{B})$, proving (1) of the theorem (another proof).

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On residual varieties of groups

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Abstract. The concept of residuation in varieties introduced by Neumann [5] and reoriented and developed by Jónsson is further studied in this paper.

Keywords. Groups; residual varieties; residuation

1. Introduction

The idea of residuation, well-known in ideal-theory, was explicitly studied in varieties of groups, by Jónsson [4]. Since multiplication of varieties is monotonic, associative, right distributive and right cancellative, there arise two types of residuations, left and right.

We denote varieties by bold latin letters, the phrase 'subvariety of' by '$\leqslant$', left residuation by (\) and right residuation by (/). Rest of the notations and symbols are as in Neumann [6].

Left residual variety $\mathbf{U}\backslash\mathbf{V}$; of the variety $\mathbf{U}$ with regard to the variety $\mathbf{V}$, is defined as the smallest variety $\mathbf{S}$ such that $\mathbf{U} \leqslant \mathbf{SV}$. In other words if $\mathbf{U} = \text{var } A$ and V denotes the set of laws defining $\mathbf{V}$ then $\mathbf{U}\backslash\mathbf{V} = \text{var } V(A)$.

Dually, right residual variety $\mathbf{U}/\mathbf{V}$; of the variety $\mathbf{U}$ with regard to the variety $\mathbf{V} \leqslant \mathbf{U}$ is defined as the largest variety $\mathbf{L}$ such that $\mathbf{U} \geqslant \mathbf{LV}$. However, existence of group B such that $\mathbf{U}/\mathbf{V} = \text{var } B$, is not known.

2. On Jónsson's results

First we mention Jónsson's results for left residuation proved in [4]:

Lemma 2.*A* For any varieties $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{S}$ of groups

$\mathbf{U}\backslash\mathbf{V} \leqslant \mathbf{S}$ iff $\mathbf{U} \leqslant \mathbf{SV}$.

Lemma 2.*B* For any varieties $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{V}'$ of groups

if $\mathbf{V} \leqslant \mathbf{V}'$ then $\mathbf{U}\backslash\mathbf{V} \geqslant \mathbf{U}\backslash\mathbf{V}'$.

Lemma 2.*C* For any varieties $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ of groups if

$\mathbf{W} \neq \mathbf{D}$ then $(\mathbf{UW})\backslash(\mathbf{VW}) = \mathbf{U}\backslash\mathbf{V}$.

Theorem 2.*A* If **U**, **V** and **W** are varieties of groups and

$\mathbf{V} \not\leqslant \mathbf{W}$ then $(\mathbf{UV})\backslash\mathbf{W} = \mathbf{U}(\mathbf{V}\backslash\mathbf{W})$.

Theorem 2.*B* If **U**, **V** and **W** are varieties of groups and

$\mathbf{V} \leqslant \mathbf{W}$ then $(\mathbf{UV})\backslash\mathbf{W} = \mathbf{U}\backslash(\mathbf{W}/\mathbf{V})$.

Now we give the dual of these results, the proofs of which are left to the reader:-

Lemma 2.*A'* $\mathbf{U}/\mathbf{V} \geqslant \mathbf{L}$ iff $\mathbf{U} \geqslant \mathbf{LV}$.

Lemma 2.*B'* If $\mathbf{V} \leqslant \mathbf{V}' \leqslant \mathbf{U}$ then $\mathbf{U}/\mathbf{V} \geqslant \mathbf{U}/\mathbf{V}'$.

Lemma 2.*C'* For any varieties **U**, **V** and **W** of groups

if $\mathbf{W} \neq \mathbf{D}$ then $(\mathbf{UW})/(\mathbf{VW}) = \mathbf{U}/\mathbf{V}$,

Theorem 2.*A'* If **U**, **V** and **W** are varieties of groups and

$\mathbf{W} \leqslant \mathbf{V}$ then $(\mathbf{UV})/\mathbf{W} \geqslant \mathbf{U}\,(\mathbf{V}/\mathbf{W})$.

Proof: Apply 21.24 [6] and definition of right residuation.

Remark 2.1 It is a conjecture that reverse inclusion does not hold.

Theorem 2.*B'* For any varieties **U**, **V** and **W** where $\mathbf{W} \leqslant \mathbf{UV}$, $(\mathbf{UV})/\mathbf{W} \geqslant \mathbf{U}/(\mathbf{W}\backslash\mathbf{V})$.

Proof: It is immediate by using 21.21 [6] and definition of right residuation.
Counter example to the reverse inclusion of theorem 2.B' is given as follows:-

Example 2.*B''*: Let $\mathbf{V} = \mathbf{A}_3\,\mathbf{A}_2$; $\mathbf{W} = \mathbf{A}_2$ then $\mathbf{W}\backslash\mathbf{V} = \mathbf{E}$ and $\mathbf{U}/(\mathbf{W}\backslash\mathbf{V}) = \mathbf{U}$ for any **U** but $(\mathbf{UV})/\mathbf{W} = \mathbf{UA}_3 \not\leqslant \mathbf{U}$.

3. Left residuation

We give some results of moderate depth for left residual varieties of groups.

Lemma 3.1 (i) For any varieties **U** and **V**; $\mathbf{U} \leqslant \mathbf{V}$ iff $\mathbf{U}\backslash\mathbf{V} = \mathbf{E}$
(ii) If $\mathbf{U} \leqslant \mathbf{U}'$ and **V** is some other variety then $\mathbf{U}\backslash\mathbf{V} \leqslant \mathbf{U}'\backslash\mathbf{V}$

Proposition 3.2 For any varieties **U** and $\mathbf{V} \neq D$,

(i) $\mathbf{U}\backslash\mathbf{V} \leqslant \mathbf{U}$

(ii) $(\mathbf{UV})\backslash\mathbf{V} = \mathbf{U}$

(iii) $(\mathbf{U}\backslash\mathbf{V})\cdot\mathbf{V} = \mathbf{U}$ iff $\exists$ a variety **S** such that $\mathbf{U} = \mathbf{SV}$

Proof: Proof of (i) is trivial. Argue with definition of left residual variety and 21.21 (Neumann [6]), to prove (ii) and (iii).

Proposition 3.3 For any varieties $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$,

(i) $\mathbf{U}\backslash\mathbf{V} = \mathbf{U}\backslash(\mathbf{U}\wedge\mathbf{V})$

(ii) $\mathbf{U}\backslash(\mathbf{VW}) = (\mathbf{U}\backslash\mathbf{W})\backslash\mathbf{V}$

(iii) $(\mathbf{U}\vee\mathbf{V})\backslash\mathbf{W} = (\mathbf{U}\backslash\mathbf{W})\vee(\mathbf{V}\backslash\mathbf{W})$

Proof: Use lemma 2.B and definition of left residual variety for (i). Parts (ii) and (iii) follow easily by definition.

Proposition 3.4 If $\mathbf{U}$ is a Schreier variety such that

$$\mathbf{U} \not\leqslant \mathbf{V} \text{ then } \mathbf{U}\backslash\mathbf{V} = \mathbf{U}$$

Proof: By 43.6 of Neumann [6], $\mathbf{U} = \mathbf{D}, \mathbf{A}_p$ or $\mathbf{A}$ which have their usual meanings. First possibility is trivial. If $\mathbf{U} = \mathbf{A}_p$ then $\mathbf{S} \geqslant \mathbf{A}_p$ but minimality of $\mathbf{A}_p$ shows $\mathbf{S} = \mathbf{A}_p$ (because $\mathbf{U} \not\leqslant \mathbf{V} \Rightarrow \mathbf{S} \neq \mathbf{E}$). Again if $\mathbf{U} = \mathbf{A}$ then $\mathbf{A} \not\leqslant \mathbf{V}$, $\mathbf{V}$ has finite exponent. If $\mathbf{S}$ had finite exponent so would $\mathbf{SV}$. Thus $\mathbf{A} \leqslant \mathbf{SV} \Rightarrow \mathbf{S} \geqslant \mathbf{A}$. But $\mathbf{S}$ is least, hence $\mathbf{S} = \mathbf{A}$.

Proposition 3.5 For any varieties $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$

(i) $\mathbf{U}\backslash(\mathbf{V}\vee\mathbf{W}) \leqslant (\mathbf{U}\backslash\mathbf{V})\wedge(\mathbf{U}\backslash\mathbf{W})$

(ii) $(\mathbf{U}\wedge\mathbf{V})\backslash\mathbf{W} \leqslant (\mathbf{U}\backslash\mathbf{W})\wedge(\mathbf{V}\backslash\mathbf{W})$

(iii) $\mathbf{U}\backslash(\mathbf{V}\wedge\mathbf{W}) \geqslant (\mathbf{U}\backslash\mathbf{V})\vee(\mathbf{U}\backslash\mathbf{W})$

Proof: Use lemma 2.B and definition of left residuation to prove (i) and (iii). Part (ii) is easily proved by definition.

The counter examples for reverse inclusion of proposition 3.5(i), 3.5(ii) and 3.5(iii) are given below as examples 3.1, 3.2 and 3.3 respectively.

Example 3.1 Let $\mathbf{U} = \mathbf{A}_3\mathbf{A}_2\vee\mathbf{A}_2\mathbf{A}_3$; $\mathbf{V} = \mathbf{A}_2\mathbf{A}_3$ and $\mathbf{W} = \mathbf{A}_3\mathbf{A}_2$ then $\mathbf{U}\backslash(\mathbf{V}\vee\mathbf{W}) = \mathbf{E}$, $\mathbf{U}\backslash\mathbf{V} = \mathbf{A}_3\mathbf{A}_2$ and $\mathbf{U}\backslash\mathbf{W} = \mathbf{A}_2\mathbf{A}_3$. This shows that reverse inclusion is false.

Example 3.2 Let $\mathbf{U} = \mathbf{A}_2\mathbf{A}_3$; $\mathbf{V} = \mathbf{A}_2\mathbf{A}_5$; $\mathbf{W} = \mathbf{A}_2\mathbf{A}_7$, $(\mathbf{U}\wedge\mathbf{V})\backslash\mathbf{W} = \mathbf{E}$ and $(\mathbf{U}\backslash\mathbf{W})\wedge(\mathbf{V}\backslash\mathbf{W}) = \mathbf{A}_2$. Clearly reverse inclusion of 3.5(ii) is false.

Example 3.3 Let $\mathbf{U} = \mathbf{A}_7\mathbf{A}_6$; $\mathbf{V} = \mathbf{A}_2$, $\mathbf{W} = \mathbf{A}_3$, $\mathbf{V}\wedge\mathbf{W} = \mathbf{E}$; $\mathbf{U}\backslash(\mathbf{V}\wedge\mathbf{W}) = \mathbf{U}$, $\mathbf{U}\backslash\mathbf{V} = \mathbf{A}_7\mathbf{A}_3$, $\mathbf{U}\backslash\mathbf{W} = \mathbf{A}_7\mathbf{A}_2$. This falsifies the reverse inclusion of 3.5(iii)

Proposition 3.6 For any varieties $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$

(i) $[\mathbf{U},\mathbf{V}]\backslash\mathbf{W} \leqslant [\mathbf{U}\backslash\mathbf{W}, \mathbf{V}\backslash\mathbf{W}]$.

(ii) $\mathbf{U}\backslash[\mathbf{V},\mathbf{W}] \leqslant (\mathbf{U}\backslash\mathbf{V})\wedge(\mathbf{U}\backslash\mathbf{W})$.

Proof: (i) $\mathbf{U} \leqslant (\mathbf{U}\backslash\mathbf{W})\ \mathbf{W}$ and $\mathbf{V} \leqslant (\mathbf{V}\backslash\mathbf{W})\ \mathbf{W}$ imply together $[\mathbf{U}, \mathbf{V}] \leqslant [\mathbf{U}\backslash\mathbf{W}, \mathbf{V}\backslash\mathbf{W}]\ \mathbf{W}$ which asserts the required result. (ii) since $V \geqslant [V, W]$ and $W \geqslant [V, W]$, we assert $\mathbf{V} \leqslant [\mathbf{V}, \mathbf{W}]$ and $\mathbf{W} \leqslant [\mathbf{V}, \mathbf{W}]$, using lemma 2.B we have $\mathbf{U}\backslash[\mathbf{V}, \mathbf{W}] \leqslant (\mathbf{U}\backslash\mathbf{V})\wedge(\mathbf{U}\backslash\mathbf{W})$.

The counter examples for reverse inclusions of proposition 3.6(i) and 3.6(ii) are as follows:

Example 3.4 Let $\mathbf{U} = \mathbf{V} = \mathbf{A}$; $\mathbf{W} = \mathbf{M}$ then $[\mathbf{U}, \mathbf{V}]\backslash\mathbf{W} = \mathbf{E}$ but $[\mathbf{U}\backslash\mathbf{W}, \mathbf{V}\backslash\mathbf{W}] = [\mathbf{E}, \mathbf{E}] = \mathbf{A}$.

Example 3.5 Let $\mathbf{V} = \mathbf{W} = \mathbf{E}$ and $\mathbf{U} = \mathbf{A}_3$ then $[\mathbf{V}, \mathbf{W}] = \mathbf{A}$ and $\mathbf{U}\backslash[\mathbf{V}, \mathbf{W}] = \mathbf{E}$, $\mathbf{U}\backslash\mathbf{V} = \mathbf{U}\backslash\mathbf{W} = \mathbf{A}_3$. This shows that reverse inclusion is false.

Query 3.7 (i) Does $\mathbf{U}\backslash\mathbf{V} = \mathbf{U}\backslash\mathbf{W}$ imply $\mathbf{V} = \mathbf{W}$ if $\mathbf{U} \neq \mathbf{D}$?

(ii) Does $\mathbf{U}_1\backslash\mathbf{U}_2 = \mathbf{V}_1\backslash\mathbf{V}_2$ imply $\mathbf{U}_1 = \mathbf{U}_2$ or $\mathbf{V}_1 = \mathbf{V}_2$?

The answers to these queries are negative. Here are the counter examples for the above.

Example 3.6 (i) $\mathbf{U} = \mathbf{A}_2$, $\mathbf{V} = \mathbf{A}_3$, $\mathbf{W} = \mathbf{A}_5$

(ii) $\mathbf{U}_1 = \mathbf{A}_3\mathbf{A}_2$, $\mathbf{U}_2 = \mathbf{A}_2$, $\mathbf{V}_1 = \mathbf{A}_3$, $\mathbf{V}_2 = \mathbf{A}_7$

4. Right residuation

We give here the dual results of §2.

Lemma 4.1 (i) $\mathbf{U} = \mathbf{V}$ iff $\mathbf{U}/\mathbf{V} = \mathbf{E}$

(ii) If $\mathbf{V} \leqslant \mathbf{U} \leqslant \mathbf{U}'$ then $\mathbf{U}/\mathbf{V} \leqslant \mathbf{U}'/\mathbf{V}$

Proposition 4.2 For any variety $\mathbf{U}$ and $\mathbf{V}$,

(i) $\mathbf{U}/\mathbf{V} \leqslant \mathbf{U}$ for $\mathbf{V} \leqslant \mathbf{U}$

(ii) $(\mathbf{UV})/\mathbf{V} = \mathbf{U}$ for $\mathbf{V} \neq \mathbf{D}$

(iii) $(\mathbf{U}/\mathbf{V})\mathbf{V} = \mathbf{U}$ iff $\exists$ a variety $\mathbf{L}$ such that $\mathbf{U} = \mathbf{LV}$ for $\mathbf{V} \neq \mathbf{D}$

Proof: Part (i) directly follows by method of contradiction using definition of right residuation. For parts (ii) and (iii) we make use of 21.21 [6] and definition of right residuation.

Proposition 4.3 For any varieties $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$

(i) $\mathbf{U}/\mathbf{V} = (\mathbf{U} \vee \mathbf{V})/\mathbf{V}$

(ii) $\mathbf{U}/(\mathbf{VW}) = (\mathbf{U}/\mathbf{W})/\mathbf{V}$ where $\mathbf{VW} \leqslant \mathbf{U}$

(iii) $(\mathbf{U} \wedge \mathbf{V})/\mathbf{W} = (\mathbf{U}/\mathbf{W}) \wedge (\mathbf{V}/\mathbf{W})$ where $\mathbf{W} \leqslant \mathbf{U} \wedge \mathbf{V}$

Proof: (i) Since $\mathbf{V} \leqslant \mathbf{U}$ in the definition of $\mathbf{U}/\mathbf{V}$ therefore $\mathbf{U} \vee \mathbf{V} = \mathbf{U}$ and hence the result. Parts (ii) and (iii) follow by definition of right residuation.

Proposition 4.4 If $\mathbf{U}$ is a Schreier variety other than $\mathbf{D}$ and $\mathbf{V} \neq \mathbf{E}$ then $\mathbf{U}/\mathbf{V} = \mathbf{E}$

Proof: If $\mathbf{U} = \mathbf{A}_p \geqslant \mathbf{V}$ then $\mathbf{L} \leqslant \mathbf{A}_p$ but $\mathbf{A}_p$ is minimal hence $\mathbf{L} = \mathbf{E}$. If $\mathbf{U} = \mathbf{A} \geqslant \mathbf{V}$

then $\mathbf{V}$ has finite exponent. If $\mathbf{L}$ had finite exponent so would have $\mathbf{LV}$ hence $\mathbf{A} \geqslant \mathbf{LV}$ implies $\mathbf{L} \leqslant \mathbf{A}$ which further asserts $\mathbf{L} = \mathbf{E}$.

Proposition 4.5 For any varieties $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$

(i) $\mathbf{U}/(\mathbf{V} \vee \mathbf{W}) \leqslant (\mathbf{U}/\mathbf{V}) \wedge (\mathbf{U}/\mathbf{W})$ where $\mathbf{V} \vee \mathbf{W} \leqslant \mathbf{U}$

(ii) $(\mathbf{U} \vee \mathbf{V})/\mathbf{W} \geqslant (\mathbf{U}/\mathbf{W}) \vee (\mathbf{V}/\mathbf{W})$ where $\mathbf{W} \leqslant \mathbf{U} \wedge \mathbf{V}$

(iii) $\mathbf{U}/(\mathbf{V} \wedge \mathbf{W}) \geqslant (\mathbf{U}/\mathbf{V}) \vee (\mathbf{U}/\mathbf{W})$ where $\mathbf{V} \wedge \mathbf{W} \leqslant \mathbf{U}$

Proof: Parts (i) and (iii) can be proved using lemma 2.B′. Part (ii) follows by definition.

The counter examples for reverse inclusion of proposition 4.5(i), 4.5(ii) and 4.5(iii) are given below as examples 4.1, 4.2 and 4.3 respectively.

Example 4.1 Let $\mathbf{U} = \mathbf{A}_3\mathbf{A}_2 \vee \mathbf{A}_2\mathbf{A}_3$; $\mathbf{V} = \mathbf{A}_2\mathbf{A}_3$, $\mathbf{W} = \mathbf{A}_3\mathbf{A}_2$ then $\mathbf{U}/(\mathbf{V} \vee \mathbf{W}) = \mathbf{E}$; $\mathbf{U}/\mathbf{V} = \mathbf{A}_3\mathbf{A}_2$ and $\mathbf{U}/\mathbf{W} = \mathbf{A}_2\mathbf{A}_3$. This falsifies the reverse inclusion.

Example 4.2 Let $\mathbf{U} = \mathbf{A}_3\mathbf{A}_2$, $\mathbf{V} = \mathbf{A}_2\mathbf{A}_3$ and $\mathbf{W} = \mathbf{M}_6$ then $(\mathbf{U} \vee \mathbf{V})/\mathbf{W} = \mathbf{M}_6$, $\mathbf{U}/\mathbf{W} = \mathbf{V}/\mathbf{W} = \mathbf{E}$. This shows that the reverse inclusion is false.

Example 4.3 Let $\mathbf{U} = \text{var } \mathbf{A}_5$, $\mathbf{V} = \mathbf{A}_2$, $\mathbf{W} = \mathbf{A}_3$ then $\mathbf{V} \wedge \mathbf{W} = \mathbf{E}$ so that $\mathbf{U}/(\mathbf{V} \wedge \mathbf{W}) = \mathbf{U}$, $\mathbf{U}/\mathbf{V} = \mathbf{A}_{15}$, $\mathbf{U}/\mathbf{W} = \mathbf{A}_2$ and clearly $\mathbf{U} \not\leqslant \mathbf{A}_{15} \vee \mathbf{A}_2 = \mathbf{A}_{30}$ which falsifies the reverse inclusion.

Proposition 4.6 For any varieties $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$

(i) $[\mathbf{U}, \mathbf{V}]/\mathbf{W} \geqslant [\mathbf{U}/\mathbf{W}, \mathbf{V}/\mathbf{W}]$ where $\mathbf{W} \leqslant \mathbf{U} \wedge \mathbf{V}$

(ii) $\mathbf{U}/[\mathbf{V}, \mathbf{W}] \leqslant (\mathbf{U}/\mathbf{V}) \wedge (\mathbf{U}/\mathbf{W})$ where $[\mathbf{V}, \mathbf{W}] \leqslant \mathbf{U}$

Proof: (i) $\mathbf{U} \geqslant (\mathbf{U}/\mathbf{W})\ \mathbf{W}$ and $\mathbf{V} \geqslant (\mathbf{V}/\mathbf{W})\ \mathbf{W}$ imply together $[\mathbf{U}, \mathbf{V}] \geqslant [\mathbf{U}/\mathbf{W}, \mathbf{V}/\mathbf{W}]\mathbf{W}$. This shows $[\mathbf{U}, \mathbf{V}]/\mathbf{W} \geqslant [\mathbf{U}/\mathbf{W}, \mathbf{V}/\mathbf{W}]$.
(ii) $\mathbf{W} \leqslant [\mathbf{V}, \mathbf{W}] \leqslant \mathbf{U}$ and $\mathbf{V} \leqslant [\mathbf{V}, \mathbf{W}] \leqslant \mathbf{U}$ imply by lemma 2.B′ that $\mathbf{U}/\mathbf{W} \geqslant \mathbf{U}/[\mathbf{V}, \mathbf{W}]$ and $\mathbf{U}/\mathbf{V} \geqslant \mathbf{U}/[\mathbf{V}, \mathbf{W}]$. These two inequalities imply $\mathbf{U}/[\mathbf{V}, \mathbf{W}] \leqslant (\mathbf{U}/\mathbf{V}) \wedge (\mathbf{U}/\mathbf{W})$.

The counter examples for reverse inclusion of proposition 4.6(i) and 4.6(ii) are as follows:-

Example 4.4 Let $\mathbf{U} = \mathbf{A}^2 = \mathbf{V}$, $\mathbf{W} = \mathbf{A}$ then $[\mathbf{U}, \mathbf{V}]/\mathbf{W} = \mathbf{A}^3$, $[\mathbf{U}/\mathbf{W}, \mathbf{V}/\mathbf{W}] = \mathbf{A}^2$. This shows that the reverse inclusion is false.

Example 4.5 Let $\mathbf{U} = \mathbf{A}^2$, $\mathbf{V} = \mathbf{W} = \mathbf{A}$ then $\mathbf{U}/[\mathbf{V}, \mathbf{W}] = \mathbf{E}$ and $\mathbf{U}/\mathbf{V} = \mathbf{U}/\mathbf{W} = \mathbf{A}$. This makes the reverse inclusion false.

Query 4.7 (i) Does $\mathbf{U}/\mathbf{V} = \mathbf{U}/\mathbf{W}$ for $\mathbf{U} \neq \mathbf{D}$ imply $\mathbf{V} = \mathbf{W}$? (ii) Does $\mathbf{U}_1/\mathbf{U}_2 = \mathbf{V}_1/\mathbf{V}_2$ imply $\mathbf{U}_1 = \mathbf{U}_2$, $\mathbf{V}_1 = \mathbf{V}_2$?

The answers to these queries are negative. Here are the counter examples for the above.

Example 4.6 (i) $\mathbf{U} = \mathbf{A}_5\mathbf{A}$, $\mathbf{V} = \mathbf{A}_3$, $\mathbf{W} = \mathbf{A}_2$ (ii) $\mathbf{U}_1 = \mathbf{A}_3\mathbf{A}_2$, $\mathbf{U}_2 = \mathbf{A}_2$, $\mathbf{V}_1 = \mathbf{A}_3\mathbf{A}_5$, $\mathbf{V}_2 = \mathbf{A}_5$

5. Irreducible decomposition of a variety

A variety $\mathbf{V}$ is called join irreducible if $\mathbf{V} = \mathbf{W} \vee \mathbf{X}$ implies $\mathbf{V} = \mathbf{W}$ or $\mathbf{V} = \mathbf{X}$ where $\mathbf{W}$ and $\mathbf{X}$ are sub-varieties of $\mathbf{V}$. Otherwise, $\mathbf{V}$ is join reducible.

Further, a decomposition $\mathbf{U} = \mathbf{V}_1 \vee \ldots \vee \mathbf{V}_m$ is called normal if (i) $\mathbf{V}_i \not\subset v_{i \neq j} \mathbf{V}_j$ for $1 \leqslant i \leqslant m$ and (ii) all components $\mathbf{V}_i$ are non-trivial join irreducible varieties.

We prove in this section that two normal decompositions are of the same length. However, corresponding components may differ in two decompositions and such examples are available in [1], [2] and [3].

Theorem 5.1 Let $\mathbf{U}$ be any variety having two normal decompositions: $\mathbf{U} = \mathbf{V}_1 \vee \mathbf{V}_2 \vee \ldots \vee \mathbf{V}_m = \mathbf{V}'_1 \vee \ldots \vee \mathbf{V}'_n$ then $m = n$.

Proof: If $\mathbf{U} = \mathbf{E}$, there is nothing to prove, so let $\mathbf{U} \neq \mathbf{E}$ and $\mathbf{V}_1, \ldots, \mathbf{V}_m, \mathbf{V}'_1, \ldots, \mathbf{V}'_n$ be proper subvarieties of $\mathbf{U}$.

We claim that $\mathbf{V}_i$ (fixed for the time being) occurs among $\mathbf{V}'_1, \ldots, \mathbf{V}'_n$. On the contrary suppose $\mathbf{V}_i \neq \mathbf{V}'_j$ for some j. Then $\mathbf{V}'_j \backslash \mathbf{V}_i \leqslant \mathbf{V}'_j$ by proposition 3.2(i); hence $\mathbf{U} \backslash \mathbf{V}_i = (\mathbf{V}'_1 \backslash \mathbf{V}_i) \vee \ldots \vee (\mathbf{V}'_n \backslash \mathbf{V}_i)$ by proposition 3.3(iii), that is, $\mathbf{U} \backslash \mathbf{V}_i = \mathbf{V}'_1 \vee \ldots \vee \mathbf{V}'_n = \mathbf{U}$. On the other hand since decomposition $\mathbf{U} = \mathbf{V}_1 \vee \ldots \vee \mathbf{V}_m$ is normal $\mathbf{V}_i \not\subset \vee \mathbf{V}_k$ so that $\mathbf{V}_k \backslash \mathbf{V}_i \leqslant \mathbf{V}_k$, $i \neq k$ by lemma 3.2(i). Therefore $\mathbf{U} \backslash \mathbf{V}_i \leqslant \mathbf{V}_1 \vee \mathbf{V}_2 \vee \ldots \mathbf{E} \vee \ldots \vee \mathbf{V}_m = \mathbf{V}^*$. These two assertions conclude that $\mathbf{U}$ has decomposition of length $m-1$. Hence $\mathbf{V}_i$ occurs among $\mathbf{V}'_j$ for $1 \leqslant i \leqslant m$.

Next, suppose $m < n$ then $\mathbf{V}_1 \vee \ldots \vee \mathbf{V}_m = \mathbf{V}'_1 \vee \ldots \vee \mathbf{V}'_m \vee \mathbf{V}'_{m+1} \vee \ldots \vee \mathbf{V}'_n$. Since each $\mathbf{V}_i$ $(1 \leqslant i \leqslant m)$ occurs among $\mathbf{V}'_j$ $(1 \leqslant j \leqslant n)$ we have $\mathbf{V}_{\sigma(1)} \vee \ldots \vee \mathbf{V}'_{\sigma(n-m)} = \mathbf{E}$ (where $\sigma(i) = i+m$ is a permutation on $\{1,2,\ldots, n-m\}$) which is contrary to the second condition of normal decomposition. Similarly we can argue for $m > n$. This establishes $m = n$.

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A Laurent series expansion formula and its applications

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Abstract. This paper gives a certain Laurent series expansion for a generalized Rodrigues type formula. The main result finds many applications which are enumerated briefly.

Keywords. Laurent series; Lagrange's expansion formula; H-functions.

1. Introduction

The literature is full of formulas giving series expansions relating the special functions of mathematical physics and one encounters several such results involving the higher classes of special functions of one and more variables. However, the closed form expressions for such series as the Laurent series $\sum_{n=-\infty}^{\infty} f(n)$, involving the special functions do not appear frequently in literature. Indeed, during the last decade significant contributions achieved in the field of *fractional calculus* saw these series expansions emerge. One finds some of these results in the papers of Osler ([3]–[5]). Further such expansions for higher classes of special functions were obtained by Raina and Co-workers ([8]–[11]).

This paper aims at deriving a Laurent series expansion for a certain type of function (equation (1)) defined below. Several applications can be motivated from our result because of the general nature of the function involved and these are briefly pointed out in the concluding section.

For any real number a such that $0 < a \leqslant 1$, and for any arbitrary number η (real or complex), define the generalized "Rodrigues Type" formula by

$$S_n^a(\eta; z) = \frac{1}{\Gamma(an+\eta+1)} D_z^{an+\eta}\{(cz+d)^{an+\eta} z^\lambda f(z)\}, \tag{1}$$

where $\mathrm{Re}(\lambda) > -1$, $f(z)$ is analytic in the circular domain $|z| < R$, and is independent of n; c, d are constants (not both zero) and $D_z^{an+\eta}$ denotes the derivative of arbitrary order.

The derivative operator D_z^α (α arbitrary) is the condensed form of the notation ${}_0D_z^\alpha$, which corresponds to the widely used notation ${}_cD_z^\alpha$ (see Ross [6] for its full definition). Under the various restrictions stated above, the operator $D_z^{an+\eta}$ occurring in (1) is meaningful and consequently the function $S_n^a(\eta; z)$ is well defined.

2. Main result

We propose to establish the following expansion formula:

Theorem. With the various constraints on the parameters and other restrictions, let $S_n^a(\eta; z)$ be defined by (1)
Then

$$\sum_{n=-\infty}^{\infty} \binom{am+an+\eta+\gamma}{an+\eta} S_{m+n}^a(\eta+\gamma; z)t^{an+\eta}$$
$$= a^{-1}(1-ct)^{-am-\gamma-1} S_m^a(\gamma; (z+dt)/(1-ct)), \tag{2}$$

holds for all $m = 0, \pm 1, \pm 2, \ldots$; provided that $|t| = |z/d|$, where γ is any arbitrary number.

Proof. If I denotes the left-hand member of (2), then from the definition (1), we readily observe that

$$I = \sum_{n=-\infty}^{\infty} D_z^{an+am+\eta+\gamma}\{(cz+d)^{an+am+\eta+\gamma} z^{\lambda} f(z)\}$$
$$\times \frac{t^{an+\eta}}{\Gamma(an+\eta+1)\Gamma(am+\gamma+1)}. \tag{3}$$

Since $\mathrm{Re}(\lambda) > -1$, the functions in (3) are analytic in the domain $0 < |z| < R$, the derivative operator of arbitrary order involved in (3) would satisfy the law of exponents

$$D_z^{q+q'} = D_z^q D_z^{q'}, \tag{4}$$

so that I can be expressed as

$$I = \frac{1}{\Gamma(am+\gamma+1)} D_z^{am+\gamma} \sum_{n=-\infty}^{\infty} D_z^{an+\eta}\{(cz+d)^{an+\eta} z^{\lambda}(cz+d)^{am+\gamma} f(z)\}$$
$$\times \frac{t^{an+\eta}}{\Gamma(an+\eta+1)}. \tag{5}$$

By appealing to the following generalized form of the Lagrange's expansion formula which is also a consequence of the more generalized form of Osler ([4] equation (1.2); see also Lavoie *et al* [2], equation (20.5)):

$$\sum_{n=-\infty}^{\infty} D_z^{an+\eta}\{f(z)[g(z)]^{an+\eta}\} \frac{t^{an+\eta}}{\Gamma(an+\eta+1)}$$
$$= a^{-1} f(w)(1-tg(w))^{-1}, \tag{6}$$

provided that $|(w-z)/g(w)| = |z/g(0)|$, where $0 < a \leqslant 1$, γ is any arbitrary number, and w is given by

$$w = z + tg(w), \tag{7}$$

we thus obtain

$$I = \frac{(1-ct)^{-1}}{a\Gamma(am+\gamma+1)} D_z^{am+\gamma}\{w^{\lambda}(cw+d)^{am+\gamma} f(w)\}, \tag{8}$$

where

$$w = (z+dt)/(1-ct). \tag{9}$$

Noting that the function w given by (9) is analytic and univalent in the domain $0 < |z| < R$, the derivative operator in (8) is also well defined for this transformation and thus (2) follows at once by using (1).

3. Applications

At the outset, we observe that our expansion formula (2) in the special case $a = 1, \eta = \gamma = 0$ and $\lambda = 0$ corresponds to the result of Srivastava ([13], equation (6)).

Since many special functions of one or more variables can fairly easily be represented in the form (1) (see the list of such representations in the paper of Lavoie *et al* ([2], pp. 260–261), our expansion formula given in the theorem is widely applicable. We find it appropriate to obtain a particular expansion formula from (2) which would yield various special cases. To this end, let

$$f(z) = H_{P,Q}^{M,N}\left[bz^h \middle| \begin{matrix}(a_i, \alpha_i)_{1,P} \\ (b_i, \beta_i)_{1,Q}\end{matrix}\right], \quad h > 0, \tag{10}$$

in (1), where the function described above is the well-known H-function of Fox ([1], p. 408) and is defined in terms of a Mellin-Barnes type contour integral.

The expressions involved in the right side of (1) obviously suggest that we also require (for closed form expressions) in our analysis, the H-function of two variables which is an extension of the function appearing in (10). These functions being well known in literature, we skip details regarding their definition, notational format and the conditions of existence. For a full account of these details, one may refer to the paper of Srivastava and Panda ([12], pp. 265–267).

The expansion formula (2) thus in conjunction with (10) on taking $d = 1$, and using suitably the known result of Raina and Koul ([7], equation (8)), with, of course, other slight changes yields the expansion

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{t^{an+\eta}}{\Gamma(an+\eta+1)\Gamma(-an-am-\eta-\gamma)} H_{1,1:(P,Q);[1,1]}^{0,1:(M,N);[1,1]} \\
&\times\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix}(-\lambda; h, 1) & : (a_i, \alpha_i)_{1,P}; (1+an+am+\eta+\gamma, 1) \\ (-\lambda+an+am+\eta+\gamma; \mathrm{h}, 1) & : (b_i, \beta_i)_{1,Q}; (0,1)\end{matrix}\right] \\
&= a^{-1}(1-yt)^{-\lambda-1} \frac{(1+t)^{\lambda-am-\gamma}}{\Gamma(-am-\gamma)} H_{1,1:(P,Q);[1,1]}^{0,1:(M,N)[1,1]} \\
&\times\left[\begin{matrix} x((1+t)/(1-yt))^h \\ y((1+t)/(1-yt)) \end{matrix}\middle|\begin{matrix}(-\lambda; h, 1) & : (a_i, \alpha_i)_{1,P}; (1+am+\gamma, 1) \\ (-\lambda+am+\gamma; h, 1) & : (b_i, \beta_i)_{1,Q}; (0,1)\end{matrix}\right],
\end{aligned} \tag{11}$$

valid for $|t| = 1$; provided that $h > 0$, $\mathrm{Re}[\lambda + h(b_i/\beta_i)] > -1$, $i = 1, \ldots, M$, $\sum_{i}^{Q} \beta_i - \sum_{1}^{P} \alpha_i > 0$ and $|\arg x| < \frac{1}{2}A\pi$, $|\arg y| < \pi$,

where $$A = \sum_{1}^{M} \beta_i - \sum_{M+1}^{Q} \beta_i + \sum_{1}^{N} \alpha_i - \sum_{N+1}^{P} \alpha_i > 0.$$

An interesting special case of (11) worth noting is the expansion for the H-function of Fox obtained recently by Raina ([9], equation (2.7)), which occurs (with one of its conditions corrected) when we formally let $y \to 0$, set $m = \gamma = 0$ and make other slight alterations.

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A note on means and proximate type of entire Dirichlet series

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Abstract. Proximate type is defined and constructed with the help of means under certain conditions for an entire Dirichlet series with index pair (p, q).

Keywords. Index-pair; (p, q)-order; (p, q)-type; (p, q)-growth; proximate type; Dirichlet series.

1. Introduction

Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, where $s = \sigma + it$, $0 \leqslant \lambda_1 < \lambda_2 < \ldots \lambda_n \to \infty$ as $n \to \infty$, be an entire Dirichlet series. Set $M(\sigma) \equiv M(\sigma, f) = \max_{t \in R} |f(\sigma + it)|$. $M(\sigma)$ is called the maximum modulus of $f(s)$. The concept of (p, q)-order, lower (p, q)-order, (p, q)-type and lower (p, q)-type of entire function $f(s)$ having index-pair (p, q), $p \geqslant (q+1) \geqslant 1$, was introduced by Juneja *et al* ([2], [3]). Thus $f(s)$ is said to be of (p, q)-order ρ if

$$\lim_{\sigma \to \infty} \sup \frac{\log^{[p]} M(\sigma)}{\log^{[q]} \sigma} = \rho(p, q) \equiv \rho \tag{1}$$

Further, if $b < \rho < \infty$ then $f(s)$ is said to be of (p, q)-type τ and lower (p, q)-type ν if

$$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(\sigma)}{(\log^{[q-1]} \sigma)^{\rho}} = \begin{matrix} \tau(p, q) \equiv \tau \\ \nu(p, q) \equiv \nu \end{matrix} \tag{2}$$

where $b = 1$ if $p = q+1$, $b = 0$ if $p > q+1$ and $\log^{[p]} x$ stands for the pth iterate of $\log x$.

Definition. An entire function $f(s)$ is said to be of perfectly regular (p, q)-growth if and only if $0 < \nu = \tau < \infty$.

For $0 < \delta < \infty$, the mean values of $f(s)$ are defined as [6]

$$I_{\delta}(\sigma) \equiv I_{\delta}(\sigma, f) = \left\{ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^{\delta} \, dt \right\}^{1/\delta} \tag{3}$$

For an entire Dirichlet series of (p, q)-order ρ, (p, q)-type τ and lower (p, q)-type ν, following Kamthan [5], we can easily prove that

$$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} I_{\delta}(\sigma)}{(\log^{[q-1]} \sigma)^{\rho}} = \begin{matrix} \tau \\ \nu \end{matrix} \tag{4}$$

The notion of proximate order for entire Dirichlet series was first introduced by Balaguer [1]. Later, Kamthan [4] also studied the properties of proximate order.

Kasana and Srivastava [7] extended the notion of proximate order to entire Dirichlet series of index-pair (p, q). In this paper we define and construct the proximate type for entire Dirichlet series with index-pair (p, q) using the mean values $I_\delta(\sigma)$.

Definition. A real valued positive function $\tau(\sigma)$ defined on $[\sigma_0, \infty)$, $\sigma_0 > 0$, is said to be a proximate type of an entire Dirichlet series with index-pair (p, q), (p, q)-order $\rho (b < \rho < \infty)$ and (p, q)-type $\tau (0 < \tau < \infty)$, if for a given constant $a (0 < a < \infty)$, $\tau(\sigma)$ satisfies the following conditions:

(i) $\tau(\sigma)$ is continuous and piecewise differentiable for $\sigma > \sigma_0$;

(ii) $\tau(\sigma) \to \tau$ as $\sigma \to \infty$,

(iii) $\Lambda_{[q-1]}(\sigma)\tau'(\sigma) \to 0$ as $\sigma \to \infty$ where $\tau'(\sigma)$ is interpreted as $\tau'(\sigma-)$ or $\tau'(\sigma+)$ whenever these are unequal, and

(iv) $$\lim_{\sigma\to\infty} \sup \frac{\log^{[p-2]} M(\sigma)}{\exp\{(\log^{[q-1]}\sigma)^\rho \tau(\sigma)\}} = a$$

where $\Lambda_{[q]}(x) = \prod_{i=0}^{q} \log^{[q]} x$.

We now prove the following result.

Theorem 1. Let $f(s)$ be an entire function with index-pair (p, q), (p, q)-order $\rho (b < \rho < \infty)$, (p, q)-type τ and lower (p, q)-type ν, $0 < \nu \leqslant \tau < \infty$. Then

$$\lim_{\sigma\to\infty} \inf \frac{\Lambda_{[q-2]}(\sigma) I'_\delta(\sigma)}{(\log^{[q-1]}\sigma)^{\rho-1} \Lambda_{[p-2]}(I_\delta(\sigma))} \leqslant \rho\nu \leqslant \rho\tau$$

$$\leqslant \lim_{\sigma\to\infty} \sup \frac{\Lambda_{[q-2]}(\sigma) I'_\delta(\sigma)}{(\log^{[q-1]}\sigma)^{\rho-1} \Lambda_{[p-2]}(I_\delta(\sigma))} \tag{5}$$

Proof. Let

$$\lim_{\sigma\to\infty} \inf \frac{I'_\delta(\sigma)\Lambda_{[q-2]}(\sigma)}{\rho\Lambda_{[p-2]}(I_\delta(\sigma))(\log^{[q-1]}\sigma)^{\rho-1}} = A.$$

For $A = 0$, the inequality in (5) is trivially true. Let $A > 0$. Then for a given $\varepsilon > 0$ and $\sigma > \sigma_0$,

$$\frac{I'_\delta(\sigma)}{\Lambda_{[p-2]}(I_\delta(\sigma))} > \frac{\rho(A-\varepsilon)(\log^{[q-1]}\sigma)^{\rho-1}}{\Lambda_{[q-2]}(\sigma)}.$$

Integrating both sides of the above inequality from σ_0 to σ, we get

$$\log^{[p-1]} I_\delta(\sigma) + 0(1) > (A-\varepsilon)(\log^{[q-1]}\sigma)^\rho.$$

Dividing by $(\log^{[q-1]}\sigma)^\rho$ and proceeding to limits we get

$$\lim_{\sigma\to\infty} \inf \frac{\log^{[p-1]} I_\delta(\sigma)}{(\log^{[q-1]}\sigma)^\rho} \geqslant A.$$

In view of (4) and the above inequality, the left side inequality of (5) is established. Similarly the right side inequality can be proved.

Theorem 2. Let $f(s)$ be an entire function of (p, q)-order ρ and (p, q)-type $\tau (0 < \tau < \infty)$. If the limit exists in (5) then $[\log(a^{-1}\log^{[p-2]} I_\delta(\sigma))]/(\log^{[q-1]}\sigma)^\rho$ is a proximate

type of $f(s)$.

Proof. For a given constant $a\,(0 < a < \infty)$, let

$$\phi(\sigma) = \frac{\log\{a^{-1}\log^{[p-2]}I_\delta(\sigma)\}}{(\log^{[q-1]}\sigma)^\rho} \tag{6}$$

Then the function $\phi(\sigma)$ is obviously continuous for $\sigma > \sigma_0$. Since $I_\delta(\sigma)$ is an almost everywhere differentiable function of σ with an increasing derivative, $\log^{[p-1]}I_\delta(\sigma)$ is differentiable almost everywhere. Consequently, $\phi(\sigma)$ is piecewise differentiable.

By assumption, the limit in (5) exists and this implies that $f(s)$ is of perfectly regular (p, q)-growth. Moreover, we have

$$\lim_{\sigma\to\infty}\phi(\sigma) = \tau.$$

Differentiating (6) with respect to σ, we get

$$\phi'(\sigma) = \frac{-\rho\log(a^{-1}\log^{[p-2]}I_\delta(\sigma))}{\Lambda_{[q-2]}(\sigma)(\log^{[q-1]}\sigma)^{\rho+1}} + \frac{I'_\delta(\sigma)}{\Lambda_{[p-2]}(I_\delta(\sigma))(\log^{[q-1]}\sigma)^\rho}.$$

Hence

$$\Lambda_{[q-1]}(\sigma)\phi'(\sigma) = \frac{I'_\delta(\sigma)\Lambda_{[q-2]}(\sigma)}{\Lambda_{[p-2]}(I_\delta(\sigma))(\log^{[q-1]}\sigma)^{\rho-1}} - \frac{\rho\log^{[p-1]}I_\delta(\sigma)}{(\log^{[q-1]}\sigma)^\rho} + o(1)$$

Using the hypothesis of the theorem and (4), we note that

$$\lim_{\sigma\to\infty}\Lambda_{[q-1]}(\sigma)\phi'(\sigma) = 0.$$

From the definition of $\phi(\sigma)$, we at once have

$$\limsup_{\sigma\to\infty}\frac{\log^{[p-2]}I_\delta(\sigma)}{\exp\{(\log^{[q-1]}\sigma)^\rho\phi(\sigma)\}} = a$$

It follows that $\phi(\sigma)$ satisfies (i) through (iv). Hence $\phi(\sigma)$ is a proximate type of $f(s)$. This proves Theorem 2.

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ERRATA

"Formula for primes, twinprimes, number of primes and number of twinprimes" by A Venugopalan, *Proc. Indian Acad. Sci. (Math. Sci.)* **92** pp. 49–52 September 1983

Read: Q at stage (1) for Q_i

$K = n$ at stage (3) for $K = n - \sum_{i=1}^{n} 1/p_i$

Read: KQ for $[K]Q \prod_{i=1}^{n} (1 - (1/p_i))$ wherever it occurs.

Read: $\sum_{a_n=1}^{(p_n-1)}$ at stage (13) for $\sum_{b=1}^{(p_n-1)}$

Read: summation limits as: $\sum_{b=0}^{n}$ wherever it appears as $\sum_{b=1}^{n}$

$1/2\pi$ for every $1/4\pi$ appearing on p. 52

Remark 4, Remark 5 for Remark 5, Remark 6 on p. 52.

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On the Rogers-Ramanujan continued fraction

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Abstract. In the "Lost" note book, Ramanujan had stated a large number of results regarding evaluation of his continued fraction

$$R(\tau)=\frac{\exp 2\pi i\tau/5}{1+}\,\frac{\exp(2\pi i\tau)}{1+}\,\frac{\exp(4\pi i\tau)}{1+}\cdots$$

for certain values of τ. It is shown that all these results and many more have their source in the Kronecker limit formula.

Keywords. Continued fractions; Kronecker limit formula; Dirichlet series.

1. The continued fraction

$$\frac{1}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\cdots,\ |x|<1$$

was introduced by Rogers [7] in his work on expansions of infinite products. He proved that

$$\frac{1}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\cdots=\prod_{n=1}^{\infty}\frac{(1-x^{5n-1})(1-x^{5n-4})}{(1-x^{5n-2})(1-x^{5n-3})}.$$

Nearly twenty years later, in 1912, Ramanujan rediscovered this continued fraction and evaluated it for various values of x. If we put $\tau=x+iy$, x, y real and $y>0$ and put

$$R(\tau)=\frac{\exp(2\pi i\tau/5)}{1+}\,\frac{\exp(2\pi i\tau)}{1+}\,\frac{\exp(4\pi i\tau)}{1+}\cdots, \tag{1}$$

then Ramanujan proved

$$\frac{1}{R(\tau)}-1-R(\tau)=\frac{\eta(\tau/5)}{\eta(5\tau)}, \tag{2}$$

where $\eta(\tau)$ is Dedekind's modular form

$$\eta(\tau)=\exp(\pi i\tau/12)\prod_{n=1}^{\infty}(1-\exp(2n\pi i\tau)).$$

Ramanujan's proof of (2) is found in his unpublished manuscripts in the Oxford

Mathematical Library. He also showed that

$$\left(\frac{1}{R(\tau)}\right)^5 - 11 - (R(\tau))^5 = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6. \tag{3}$$

(See [4]). From (2) it follows that $R(\tau)$ is an elliptic modular function. Indeed it belongs to the principal congruence subgroup

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } 5)$$

of the modular group [[2], p. 383]. If we put $(\tau+5)/2$ instead of τ we then have

$$-S(\tau) = R\left(\frac{\tau+5}{2}\right) = -\left(\frac{\exp(\pi i\tau/5)}{1-}\,\frac{\exp(\pi i\tau)}{1+}\,\frac{\exp(2\pi i\tau)}{1-}\cdots\right). \tag{4}$$

Analogous to (2) and (3) one has

$$\frac{1}{S(\tau)} + 1 - S(\tau) = \frac{\eta(\tau/5)f(\tau/5)}{\eta(5\tau)f(5\tau)}, \tag{5}$$

$$\left(\frac{1}{S(\tau)}\right)^5 + 11 - (S(\tau))^5 = \left(\frac{\eta(\tau)f(\tau)}{\eta(5\tau)f(5\tau)}\right)^6, \tag{6}$$

where $f(\tau)$ is Schläfli's modular function

$$f(\tau) = \exp[-(\pi i/24)]\,\frac{\eta\left(\frac{1+\tau}{2}\right)}{\eta(\tau)}. \tag{7}$$

It is well-known that $f(\tau)$ and $\eta(\tau)$ satisfy the transformation formulae

$$f(-1/\tau) = f(\tau);\ \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau), \tag{8}$$

where $(-i\tau)^{1/2}$ is positive for τ on the imaginary positive axis.

In his Note books [[5] Vol. II, p. 204] Ramanujan stated the two following formulae

$$\left(\frac{\sqrt{5}+1}{2} + R(i\alpha)\right)\left(\frac{\sqrt{5}+1}{2} + R(i\beta)\right) = \sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)$$

$$\left(\frac{\sqrt{5}-1}{2} + S(i\alpha)\right)\left(\frac{\sqrt{5}-1}{2} + S(i\beta)\right) = \sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right) \tag{9}$$

where $\alpha > 0$, $\beta > 0$ and $\alpha\beta = 1$. A proof of this was given by Watson [9]. Recently we gave [3] a simple proof of these and in addition proved two new formulae

$$\left[\left(\frac{\sqrt{5}+1}{2}\right)^5 + (R(i\alpha))^5\right]\left[\left(\frac{\sqrt{5}+1}{2}\right)^5 + (R(i\beta))^5\right] = 5\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^5 \tag{10}$$

$$\left[\left(\frac{\sqrt{5}-1}{2}\right)^5 + (S(i\alpha))^5\right]\left[\left(\frac{\sqrt{5}-1}{2}\right)^5 + (S(i\beta))^5\right] = 5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5 \tag{11}$$

whenever $\alpha > 0$, $\beta > 0$ and $\alpha\beta = 1/5$.

Ramanujan stated (10) in his unpublished manuscripts but (11) is not to be found anywhere though Ramanujan must certainly have known it.

In the "Lost" Note book Ramanujan [6] states a number of results on the continued

fractions $R(\tau)$ and $S(\tau)$. Indeed on one page he states:

$$\text{“}F(x) = \frac{x^{1/5}}{1-}\,\frac{x}{1+}\,\frac{x^2}{1-}\,\frac{x^3}{1+}\cdots$$

$$F(\exp(-\pi/\sqrt{5})) = \left\{-\left(\frac{\sqrt{5}-1}{2}\right)^5 + \left[1+\left(\frac{\sqrt{5}-1}{2}\right)^{10}\right]^{1/2}\right\}^{1/5}$$

$$\begin{cases} F(\exp(-\pi/\sqrt{15})) = \\ F(\exp(-\pi\sqrt{3/5})) = \end{cases}$$

$$F(\exp(-\pi/5)) =$$

$$\begin{cases} F(\exp(-\pi/\sqrt{35})) = [5\sqrt{5}-7+(35(5-2\sqrt{5}))^{1/2}]^{1/5}) \\ F(\exp(-\pi\sqrt{7/5})) = [-5\sqrt{5}-7+(35(5+2\sqrt{5}))^{1/2}]^{1/5} \end{cases}$$

$$\begin{cases} F(\exp(-\pi/\sqrt{45})) = \\ F(\exp(-\pi\sqrt{9/5})) = \end{cases}$$

. . . . ”.

It is obvious from the way he brackets the results, that they are related to (11) which Ramanujan must have known.

Our object in this note is not only to prove these results and other results of Ramanujan's stated in the Note books, but also to show that the real source of such results is to be found in the Kronecker limit formula. We thus obtain many other results not given by Ramanujan.

2. Clearly $S(i\alpha) = F(\exp(-\pi\alpha))$, $\alpha > 0$.

We shall therefore first deal with the two simple cases given by Ramanujan

$$\alpha = \frac{1}{\sqrt{5}}, \quad \alpha = \frac{1}{5}.$$

If we put $\alpha = \beta = \dfrac{1}{\sqrt{5}}$ in (11) we at once get

$$\left(\frac{\sqrt{5}-1}{2}\right)^5 + (S(i/\sqrt{5}))^5 = \left[5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5\right]^{1/2}$$

which shows

$$\left[\frac{\exp(-\pi/5\sqrt{5})}{1-}\,\frac{\exp(-\pi/\sqrt{5})}{1+}\,\frac{\exp(-2\pi/\sqrt{5})}{1-}\cdots\right]^5 = -\left(\frac{\sqrt{5}-1}{2}\right)^5 +$$

$$+\left\{\left[\left(\frac{\sqrt{5}+1}{2}\right)^5 + \left(\frac{\sqrt{5}-1}{2}\right)^5\right]\left(\frac{\sqrt{5}-1}{2}\right)^5\right\}^{1/2}.$$

Hence we have

$$\frac{\exp(-\pi/5\sqrt{5})}{1-}\,\frac{\exp(-\pi/\sqrt{5})}{1+}\,\frac{\exp(-2\pi/\sqrt{5})}{1-}\cdots$$
$$=\left[-\left(\frac{\sqrt{5}-1}{2}\right)^5+\left(1+\left(\frac{\sqrt{5}-1}{2}\right)^{10}\right)^{1/2}\right]^{1/5} \tag{12}$$

Next let us take $\alpha=\beta=1$ in (9). Then

$$S(i)=\left(\frac{5-\sqrt{5}}{2}\right)^{1/2}-\frac{\sqrt{5}-1}{2}$$

In (11) let us take $\alpha=1$, $\beta=1/5$. Then

$$(S(i/5))^5=\frac{5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5}{\left(\frac{\sqrt{5}-1}{2}\right)^5+\left\{\left(\frac{5-\sqrt{5}}{2}\right)^{1/2}-\left(\frac{\sqrt{5}-1}{2}\right)\right\}^5}-\left(\frac{\sqrt{5}-1}{2}\right)^5.$$

We could simplify this and obtain finally

$$\frac{\exp(-\pi/25)}{1-}\,\frac{\exp(-\pi/5)}{1+}\,\frac{\exp(-2\pi/5)}{1-}\cdots=\left\{\left(\frac{\sqrt{5}+1}{2}\right)^5\right.$$
$$\left.-\left(\frac{\sqrt{5}-1}{2}\right)^5\left[\left(\frac{5+\sqrt{5}}{2}\right)^{1/2}-1\right]^5\right\}\Big/1+\left[\left(\frac{5+\sqrt{5}}{2}\right)^{1/2}-1\right]^5. \tag{13}$$

3. Let $n>0$ be an odd integer and let us take $\tau=i/\sqrt{5n}$ in (6). Using (8) we obtain

$$\left[S\left(\frac{i}{\sqrt{5n}}\right)\right]^{-5}+11-\left[S\left(\frac{i}{\sqrt{5n}}\right)\right]^5=5^3\left[\frac{\eta\left(\frac{1+\sqrt{-5n}}{2}\right)}{\eta\left(\frac{1+\sqrt{-5n}/5}{2}\right)}\right]^6. \tag{14}$$

Let us now assume that $-5n$ is a fundamental discriminant *i.e.* that the imaginary quadratic field $F=Q(\sqrt{-5n})$ has discriminant $-5n$. Furthermore let every genus of ideal classes of F have only one class in it. Such discriminants are listed in [1].

Let us assume for simplicity that F has class number 2. From the tables in [1] we see that

$$n=3,\ 7,\ 23,\ 47 \tag{15}$$

are numbers with this property. The two ideal classes in $F=Q(\sqrt{-5n})$ will be represented by the ideals A and B with minimal bases

$$A=\left(1,\frac{1+\sqrt{-5n}}{2}\right),\ B=\left(1,\frac{1+\sqrt{-5n}/5}{2}\right) \tag{16}$$

and with norms 1 and 1/5 respectively. Since the class number is 2, there is only one non-trivial genus character χ and this corresponds to the decomposition

$$-5n = -n \cdot 5$$

(Note that since $-5n \equiv 1 \pmod 4$, $-n \equiv 1 \pmod 4$). The L-series of F corresponding to this character has the property (Siegel [8])

$$L_F(1, \chi) = \frac{4\pi}{u_n \cdot \sqrt{5n}} \cdot h(-n) \cdot \log \frac{\sqrt{5}+1}{2} \tag{17}$$

where u_n is the number of roots of unity in F and $h(-n)$ is the class number of $Q(\sqrt{-n})$.
The Kronecker limit formula gives ([18] p. 71)

$$\log \left| \eta\left(\frac{1+\sqrt{-5n}}{2}\right)\right|^2 - \log \frac{1}{\sqrt{5}} \left| \eta\left(\frac{1+\sqrt{-5n}/5}{2}\right)\right|^2$$

$$= -\frac{2h(-n)}{u_n} \log \frac{\sqrt{5}+1}{2}. \tag{18}$$

Combining this with (14) we have the

THEOREM 1. If $-5n$ is the discriminant of the imaginary quadratic field $F = Q(\sqrt{-5n})$, where n is odd and F has class number two, then

$$\frac{\exp(-\pi/5\sqrt{5n})}{1-} \frac{\exp(-\pi/\sqrt{5n})}{1+} \frac{\exp(-2\pi/\sqrt{5n})}{1-} \cdots$$

$$= \left\{\frac{1}{2}(\alpha + (\alpha^2+4)^{1/2})\right\}^{1/5}$$

where $$\alpha = 11 - 5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^{6h(-n)/u_n}.$$

$h(-n)$ is the class number of $Q(\sqrt{-n})$ and u_n the number of roots of unity in it.

We can obtain a companion result either by using (11) or by proceeding as for the previous theorem:

THEOREM 1′: Under the same conditions as for theorem 1,

$$\frac{\exp[(-\pi/5)(\sqrt{n/5})]}{1-} \frac{\exp[-\pi(\sqrt{n/5})]}{1+} \frac{\exp[-2\pi(\sqrt{n/5})]}{1-} \cdots$$

$$= \frac{1}{2}(\beta + (\beta^2+4)^{1/2})^{1/5} \tag{20}$$

where $$\beta = 11 - 5\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^{6h(-n)/u_n}.$$

We now give some examples which include Ramanujan's given in §1.

(i) Let $n = 3$. Then $h(-3) = 1$ and $u_3 = 6$. This gives

$$\frac{\exp(-\pi/5\sqrt{15})}{1-}\,\frac{\exp(-\pi/\sqrt{15})}{1+}\,\frac{\exp(-2\pi/\sqrt{15})}{1-}\cdots$$

$$= \left[\frac{-3+5\sqrt{5}+(30(5+\sqrt{5}))^{1/2}}{4}\right]^{1/5}\cdot$$

$$\frac{\exp[(-\pi/5)(\sqrt{3/5})]}{1-}\,\frac{\exp(-\pi\sqrt{3/5})}{1+}\,\frac{\exp(-2\pi\sqrt{3/5})}{1-}\cdots$$

$$= \left[\frac{-3-5\sqrt{5}+(30(5-\sqrt{5}))^{1/2}}{4}\right]^{1/5}\cdot$$

(ii) Let now $n = 7$, then $h(-7) = 1$ and $u_7 = 2$. This gives the Ramanujan results

$$\frac{\exp(-\pi/5\cdot\sqrt{35})}{1-}\,\frac{\exp(-\pi/\sqrt{35})}{1+}\,\frac{\exp(-2\pi/\sqrt{35})}{1-} = \cdots$$

$$= [-7+5\sqrt{5}+(35(5-2\sqrt{5}))^{1/2}]^{1/5}$$

$$\frac{\exp[(-\pi/5)(\sqrt{7/5})]}{1-}\,\frac{\exp(-\pi\sqrt{7/5})}{1+}\,\frac{\exp(-2\pi\sqrt{7/5})}{1-}\cdots$$

$$= [(-7-5\sqrt{5}+(35(5+2\sqrt{5}))^{1/2}]^{1/5}$$

(iii) Let now $n = 23$ which is given in (15). Then from [1] we see that $h(-23) = 3$ and $u_{23} = 2$;

$$\frac{\exp(-\pi/5\sqrt{115})}{1-}\,\frac{\exp(-\pi/\sqrt{115})}{1+}\,\frac{\exp(-2\pi/\sqrt{115})}{1-}\cdots$$

$$= [-207+95\sqrt{5}+3((1955-874\sqrt{5})5))^{1/2}]^{1/5}$$

$$\frac{\exp[(-\pi/5)(\sqrt{23/5})]}{1-}\,\frac{\exp(-\pi\sqrt{23/5})}{1+}\,\frac{\exp(-2\pi\sqrt{23/5})}{1-}\cdots$$

$$= [-207-95\sqrt{5}+3(5(1955+874\sqrt{5}))^{1/2}]^{1/5}$$

One can consider the case where $Q(\sqrt{-5n})$, with n odd has class number 4 and with four genera, for example

$$n = 39,\ 87,\ 111,\ 119,\ 159,\ 287.$$

However we shall omit dealing with this here since an analogous situation with n even is studied in the next section from which the method would become obvious.

4. We shall consider the case of $R(i\alpha)$ where

$$\alpha = \frac{1}{\sqrt{m}},\ m > 0$$

and $10\,m$ is an idoneal number of Euler. In this case $F = Q(\sqrt{-10\,m})$ has one class in each genus. If we use (8) we have

$$\left(R\left(\frac{i}{\sqrt{10\,m}}\right)\right)^{-5} - 11 - \left(R\left(\frac{i}{\sqrt{10\,m}}\right)\right)^{5} = 5^3 \cdot \left[\frac{\eta(\sqrt{-10\,m})}{\eta(\sqrt{-10}\,m/5)}\right]^6. \quad (21)$$

Since $F = Q(\sqrt{-10\,m})$ has one class in each genus, it follows from the table [[1] (p. 426–427)], that

$$10\,m = 10,\ 30,\ 70,\ 130,\ 190,\ 210,\ 330 \quad (22)$$

We have taken only the cases where m is odd. Of these numbers 10 is the only number for which $Q(\sqrt{-10})$ has class number 2 (with of course 2 genera). The numbers 30, 70, 130 and 190 are all associated with fields having 4 genera with one class in each genus.

We shall consider only the two cases 10 and 30 which illustrate our methods.

The two classes in $F = Q(\sqrt{-10})$ are represented by the ideals

$$A = (1, \sqrt{-10}),\ B = (1, \sqrt{-10}/5)$$

with norms 1 and 1/5. There is one non-trivial genus character χ corresponding to the decomposition

$$-40 = -8.5$$

and so the L-series of F has the value

$$L_F(1, \chi) = \frac{4\pi}{2\sqrt{40}} \log\frac{\sqrt{5}+1}{2}.$$

and by the Kronecker limit formula,

$$\log\eta(\sqrt{-10})^2 - \log\frac{1}{\sqrt{5}}\eta\left(\frac{\sqrt{-10}}{5}\right)^2 = \log\frac{\sqrt{5}-1}{2}. \quad (23)$$

From (21) and (23) we obtain

$$\left(R\left(\frac{i}{\sqrt{10}}\right)\right)^{-5} - 11 - \left(R\left(\frac{i}{\sqrt{10}}\right)\right)^{5} = 5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^3.$$

This gives at once

$$\frac{\exp(-2\pi/5\sqrt{10})}{1+}\,\frac{\exp(-2\pi/\sqrt{10})}{1+}\,\frac{\exp(-4\pi/\sqrt{10})}{1+}\cdots$$
$$= [-18+5\sqrt{5}+3(\sqrt{10}(5-2\sqrt{5}))^{1/2}]^{1/5}.$$

Similarly

$$\frac{\exp[(-2\pi/5)(\sqrt{2/5})]}{1+}\,\frac{\exp(-2\pi\sqrt{2/5})}{1+}\,\frac{\exp(-4\pi\sqrt{2/5})}{1+}\cdots$$
$$= [-18-5\sqrt{5}+3(\sqrt{10}(5+2\sqrt{5}))^{1/2}]^{1/5}$$

The field $F = Q(\sqrt{-30})$ has 4 classes and 4 genera. The four classes are represented

by the ideals A, B, C, D with bases $A=(1,\sqrt{-30})$, $B=(1,\sqrt{-30}/2)$, $C=(1,\sqrt{-30}/3)$, $D=(1,\sqrt{-30}/5)$. For any character χ of the group of genera ([8] p. 62–71).

$$\left.\begin{aligned}\log F(A)+\chi(B)\log F(B)+\chi(C)\log F(C)+\chi(D)\log F(D)\\ =\frac{-\sqrt{120}}{2\pi}L_F(1,\chi)\end{aligned}\right\} \tag{24}$$

Let χ_2, χ_3 and χ_5 be the three non-trivial characters, corresponding respectively to the decompositions

$$\begin{aligned}\chi_2&:\ -120=-15.8\\ \chi_3&:\ -120=-3.40\\ \chi_5&:\ -120=-24.5\end{aligned}$$

From their definition ([8] p. 60)

$$\left.\begin{aligned}\chi_2(A)=1,\ \chi_2(B)=1,\ \chi_2(C)=-1,\ \chi_2(D)=-1\\ \chi_3(A)=1,\ \chi_3(B)=-1,\ \chi_3(C)=1,\ \chi_3(D)=-1\end{aligned}\right\} \tag{25}$$

From (24) and (25), it follows that

$$\log F(A)-\log F(D)=-\frac{\sqrt{120}}{4\pi}(L_F(1,\chi_2)+L_F(1,\chi_3)).$$

However

$$L_F(1,\chi_2)=\frac{4\pi}{2\sqrt{120}}h(-15)\,h(8)\log(\sqrt{2}+1)$$

$$L_F(1,\chi_3)=\frac{4\pi}{6\sqrt{120}}h(-3)\cdot h(40)\log(3+\sqrt{10})$$

and therefore

$$\begin{aligned}\left(R\left(\frac{i}{\sqrt{30}}\right)\right)^{-5}-11-\left(R\left(\frac{i}{\sqrt{30}}\right)\right)^{5}&=5^3\left(\frac{\eta(\sqrt{-30})}{\eta(\sqrt{-30}/5)}\right)^6\\ &=5\sqrt{5}(\sqrt{2}+1)^6(3+\sqrt{10})^2.\end{aligned} \tag{26}$$

The solution of the quadratic equation in $(R(i/\sqrt{30}))^5$ leads to the value of $R(i/\sqrt{30})$.

It is now obvious how one should proceed in other cases where $Q(\sqrt{-5n})$ or $Q(\sqrt{-10n})$ have 4 or 8 or even 16 genera with one class in each genus. It might be interesting to evaluate $R(i/\sqrt{10n})$ or $S(i/\sqrt{5n})$ in other cases related to quadratic binary forms with one class in each genus.

5. We conclude this note by proving another of Ramanujan's statements which is however incomplete and which is given in the Note books ([5] Vol 1, p. 311). This concerns the evaluation of $R(2i\alpha)$ for $\alpha = 1, 2, 4$. Let us take the continued fraction

$$R(2i) = \frac{\exp(-4\pi/5)}{1+} \frac{\exp(-4\pi)}{1+} \frac{\exp(-8\pi)}{1+} \cdots$$

In order to evaluate this, observe that since by (2)

$$\frac{1}{R(2i)} - 1 - R(2i) = \frac{\eta(2i/5)}{\eta(10i)},$$

we have only to find the value of $\eta(2i/5)/\eta(10i)$. Clearly if we put

$$\frac{\eta(2i/5)}{\eta(10i)} = 2C - 1$$

then

$$R(2i) = (C^2+1)^{1/2} - C. \tag{27}$$

Let, in the usual notation of elliptic functions

$$\frac{K'}{K} = \frac{1}{5}, \frac{L'}{L} = 5$$

and k, k' and l, l' the respective associated moduli. Then

$$\frac{\eta(i/5)}{\eta(5i)} = (K/L)^{1/2} \cdot (kk'/ll')^{1/6}$$

Let us further assume that

$$\frac{K_0'}{K_0} = 2\frac{K'}{K}, \frac{L_0'}{L_0} = 2\frac{L'}{L}$$

and k_0, k_0' and l_0, l_0' the associated moduli. Then

$$\frac{\eta(2i/5)}{\eta(10i)} = (K_0/L_0)^{1/2} \cdot (k_0k_0'/l_0l_0')^{1/6}.$$

On the other hand, by the formulae of Jacobi-Legendre,

$$K_0 = \frac{1}{2}K(1+k'), L_0 = \frac{1}{2}L(1+l')$$

and

$$k_0 = \frac{1-k'}{1+k'}, k_0' = \frac{2\sqrt{k'}}{1+k'}, l_0 = \frac{1-l'}{1+l'}, l_0' = \frac{2\sqrt{l'}}{1+l'}.$$

Therefore we finally have

$$\frac{\eta(2i/5)}{\eta(10i)} = (K/L)^{1/2}(k/l)^{1/3} \cdot (l'/k')^{1/12}$$

Since from (8), $\eta(i/5) = \sqrt{5}\,\eta(5i)$, we have

$$(K/L)^{1/2} = \sqrt{5}(ll'/kk')^{1/6}$$

Substituting we finally get

$$\frac{\eta(2i/5)}{\eta(10i)} = \sqrt{5}\cdot(k/l)^{1/6}\cdot(l'/k')^{1/12}. \tag{28}$$

In order to find k, k', l and l', notice that since $K'/K = 1/5$,

$$f(i/5) = 2^{1/6}/(kk')^{1/12}.$$

On the other hand $f(i/5) = f(5i) = 2^{1/6}/(ll')^{1/12}$. From the tables of Weber and Ramanujan

$$f(5i) = (\sqrt{5}+1)/2^{3/4}$$

which gives

$$k^2 = \frac{1}{2}\left[1+\left(1-\left(\frac{\sqrt{5}-1}{2}\right)^{24}\right)^{1/2}\right] \tag{29}$$

(It is interesting to note that this result is given by Ramanujan [[5] Vol. 1, p. 287]). One can simplify this using the fact that

$$\frac{\sqrt{5}+1}{2}\cdot\frac{\sqrt{5}-1}{2} = 1.$$

Thus

$$k^2 = \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^6\left\{\left(\frac{\sqrt{5}+1}{2}\right)^6+\left[\left(\frac{\sqrt{5}+1}{2}\right)^{12}-\left(\frac{\sqrt{5}-1}{2}\right)^{12}\right]^{1/2}\right\}$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^6(9+4\sqrt{5}+12\sqrt[4]{5})$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^6(3+2\sqrt[4]{5})^2.$$

Since $k'^2 = 1-k^2$, we have

$$k'^2 = \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^6(3-2\sqrt[4]{5})^2.$$

Now $f(i/5) = f(5i)$ and so $k^2 \neq l^2$; but since $kk' = ll'$, we have $k^2 = l'^2$ and $k'^2 = l^2$. Taking k, k', l, l' positive we have

$$\frac{\eta(2i/5)}{\eta(10i)} = \sqrt{5}\cdot(k/k')^{1/4} = \sqrt{5}\left(\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}\right)^{1/4}.$$

It is easy to verify that

$$(\sqrt[4]{5}\pm 1)^4 = 2(1+\sqrt{5})(3\pm 2\sqrt[4]{5}).$$

We thus finally arrive at Ramanujan's result (not completely stated),

THEOREM 2. If $(\sqrt[4]{5}+1/\sqrt[4]{5}-1)\sqrt{5}=2C+1$, then

$$\frac{\exp(-4\pi/5)}{1+}\,\frac{\exp(-4\pi)}{1+}\,\frac{\exp(-8\pi)}{1+}\ldots=(C^2+1)^{1/2}-C.$$

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Ramanujan as a patient

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1. Introduction

In this article I attempt to give as full a picture as possible of Srinivasa Ramanujan's medical history during his time in England. The account is based largely on unpublished letters and documents housed in various libraries, and on information extracted from the records of public bodies. In my earlier paper [12] I was concerned mainly with his notebooks and manuscripts, although I did give a brief description of the nursing homes in which he was a patient, based on information available to me at the time. What might be called Ramanujan's domestic history is well documented [1, 5, 7, 8, 9, 10, 14], at any rate as regards incident, if not exact date, so that, in the interests of brevity, I recall only those events that have a direct bearing on my subject. I also take the opportunity (in §5) to amplify and extend some of the material included in [12], in the light of information more recently received, and to make a number of general comments.

The source to which most frequent reference is made is the collection of unpublished papers and documents written by, or concerning, Ramanujan (1887–1920), which were passed on by G H Hardy (1877–1947) and J E Littlewood (1885–1977) to G N Watson (1886–1965) and are now housed in the Wren Library of Trinity College, Cambridge, where at one time all four were Fellows. I have now been able to examine them in greater detail than was possible when I first dispatched them to Trinity. This collection of papers is listed under two headings, Add. Ms. 94 and Add. Ms. 100–107. For brevity they will be referred to as W94 and W100-107, respectively; thus the first references W94, 1(20) below is an abbreviation for Add. Ms. 94 1(20).

In [5, p. xxxi] Hardy states

> It was in the spring of 1917 that Ramanujan first appeared to be unwell. He went into a Nursing Home at Cambridge in the early summer, and was never out of bed for any length of time again. He was in sanatoria at Wells, at Matlock, and in London, and it was not until the autumn of 1918 that he showed any decided symptom of improvement.

I have been unable so far to determine the exact dates of Ramanujan's residence in the various medical establishments in which he was a patient between 1917 and his return to India early in 1919, nor do any official medical records appear to be extant. Quite a lot of information is available, however, from letters and other documents that survive, and these I reproduce in full in what follows.

2. Mr Ramanujan's accounts

This is the heading in [W94, 1(20)] of the accompanying list of items of expenditure and receipt. Apart from Ramanujan, Hardy and E H Neville (1889–1961), eight persons are named and I have been able to identify seven of these with a fair degree of certainty. On the assumption that the dates given are those on which the various payments were made, the list enables one to make certain deductions concerning Ramanujan's period of stay in the various institutions where he was a patient. It should be borne in mind, however, that bills were not submitted to creditors so swiftly then as now.

The sheet bears a hand-written note by Hardy stating: "This is a duplicate. The original, signed by Laurence and me, was sent to University of Madras. G H H" Laurence is, doubtless, Reginald Vere Laurence (1876–1934), an historian, who was at the time a Senior Tutor of Trinity; he was immediately junior to Hardy as a Fellow. As is made clearly by his letter of 26 May 1920 [1, p. 77], Hardy arranged with Barclays Bank [W94, 1(21)] to pay the small amount (of the order of £30) outstanding from Ramanujan's account, although this was invalidated by the latter's death (on 26 April 1920), which intervened. The settlement would have left more than £300 in the account, with further funds to come from the Trinity Fellowship and Madras Scholarship.

The third last entry under expenditure (note that the year is omitted), together with Hardy's letter of 20 September 1917 [14, p. 69] to S M Subramanian, confirms my suggestion [12, p. 95] that, after Ramanujan first fell ill, he was admitted to the Nursing Hostel in Thompson's Lane in Cambridge. This would appear to have been in May 1917; see Hardy's letter of 26 November 1918 [1, p. 76]. As some indication of the value of money at the time, it may be mentioned that the salary of the matron of the hostel for the year 1914–15 was of the order of £70; she would presumably receive in addition

Expenditure	£			*Receipts*			
Nov. 22, 1917, Dr Mutlin	3	13	11	Dec. 17, 1917, Mr Ramanujan	100	0	0
Dec. 1, 1917, Dr L Ram	14	0	0	Mar. 20, 1918, Mr Hardy	50	0	0
Dec. 24, 1917, Dr L Ram	12	18	2	July 4, 1918, Mr Ramanujan	100	0	0
Jan. 3, 1918, Matlock San.	15	18	1	Oct. 22, 1918, Mr Ramanujan	100	0	0
Mar. 21, 1918, Dr Kincaid	63	15	0	Feb. 12, 1919, Mr Ramanujan	100	0	0
Mar. 21, 1918, Capt. Wynn	21	0	0	Aug. 23, 1919, Mr Ramanujan	100	0	0
Apr. 30, 1918, Dr L Humphrey	2	2	0	Aug. 23, 1919, Mr Hardy	14	13	8
Apr. 30, 1918, Dr Wingate	25	9	0				
July 3, 1918, Dr Kincaid	87	5	10				
Nov. 2, 1918, Home Hosp. Assoc.	50	8	0				
Dec. 5, 1918, Mr Neville	2	15	0				
Dec. 8, 1918, Mr Phillips	21	0	0				
Dec. 9, 1918, Home Hosp. Assoc.	29	18	2				
Jan. 13, 1919, Mr Phillips	20	18	8				
Mar. 17, 1919, Dr Kincaid	38	18	8				
Mar. 17, Nursing Hostel	30	6	2				
Mar. 25, 1919, Mr Hardy	54	0	0				
Aug. 23, 1919 Dr Bolton	70	7	0				
	564	13	8		564	13	8

free board and lodging. In 1917 the purchasing power of the pound sterling was approximately 25 times as high as it is today (1983).

Some time before 1921 the hostel ceased to function as a nursing home and was converted by Magdalene College to students' lodgings. It was held on a long lease by Magdalene from St John's College, but this was terminated in 1969; the building was demolished in January 1981. The successor to the Nursing Hostel is the Evelyn Nursing Home in Trumpington Road, which was established in 1921.

From the sums of money involved it would appear that Ramanujan, after seeing Dr Laurence Humphry (Registered with General Medical Council 1880), came under the care of Dr William Warburton Wingate (Registered 1890). Both these gentlemen were doctors in Cambridge at the time, the latter being Honorary Medical Officer of the Cambridge and District Nursing Association.

On 20 September 1917 Ramanujan had left the Nursing Hostel and was in Cambridge [14, pp. 69–75]. Presumably it was after that date that he went to Wells. None of the doctors listed in the accounts practised in or near that city, unless the name of Dr Mutlin, whose account for £3.13.11 was paid on 22 November 1917, is a corruption of the second element of the surname of Dr David Jacob Aaron Chowry-Muthu, who was Chief Physician and Proprietor of the Mendip Hills Sanatorium at Hill Grove, near Wells in Somerset; the misspelling of Dr Humphry's name gives some support to this conjecture, and the name Mutlin does not occur in the *Medical Register* or the *Medical Directory* for that time. Chowry-Muthu ran his sanatorium for patients with tuberculosis. He was the author of several medical works, including an account of ancient Hindu medicine. The small magnitude of the fee and the lack of any other reference in the documents extant to Wells suggest that Ramanujan's sojourn in Hill Grove was brief.

From the date of the first payment to Dr Ram it would seem that Ramanujan was admitted to the Matlock House Sanatorium in Matlock, Derbyshire, some time before December 1917. It is known that he was there in June 1918 [12, p. 94], but he does not appear to have been continuously resident there between these two months; see §4. I have already given some information about Dr Frederick Kincaid, the medical superintendent of the sanatorium, and Dr Labbhu Ram, who had patients there, in my earlier article [12, pp. 94–96]; Dr Ram lived not far away from Matlock in Mexborough. The Captain Wynn mentioned in the accounts is Dr William Henry Wynn (1878–1956), who served as a Captain in the 1st Southern Division of the Royal Army Medical Corps Territorial Force during World War I. He practised as a doctor in Birmingham and was Consulting Tuberculosis Physician to Derbyshire County Council. It is likely that he examined Ramanujan while the latter was in Matlock.

Some time after the end of June 1918 Ramanujan left Matlock. Whether he returned to Cambridge before entering Fitzroy House, the hospital run by the Home Hospitals Association in Fitzroy Square, London, is not known. According to a letter dated 26 November 1918 [3, pp. 75–76; 14, pp. 76–77] from Hardy to Francis Dewsbury, the Registrar of the University of Madras, Ramanujan had been in London during the last few months and had seen several specialists. In [12, p. 95] I concluded that the letter [W94, 2(22)] to Hardy from Fitzroy House was written shortly after Ramanujan's election on 13 October 1918 to a Fellowship at Trinity College. This tallies with the dates of the two payments to the Association listed in the accounts.

As is well known, Ramanujan was also a patient in Colinette House, 2 Colinette Road, Putney, in S W London. It was from this address on 11 January 1919 that Hardy

acted as his amanuensis in a letter to the University of Madras making recommendations for the disposal of his surplus income; see [10, pp. 103–104 and plate 6]. Hardy's well-known story of the taxi-cab with number 1729 [4, p. xxxv] refers to his visit to Ramanujan in this nursing home. The house has now reverted to private use. Its proprietor in 1919 was a Samuel Mandeville Phillips. From the payments made to him one may guess that Ramanujan went first to Putney in November 1918.

I have not been able to identify the Dr Bolton to whom the last payment was made on 23 August 1919. I incline to believe that he may have been a London specialist who had Ramanujan as a patient during his last months in England. There are several such listed in the Medical Directory, but the most likely is, in my view, Dr Charles Bolton, F.R.S. (1870–1947), who was Consulting Physician to University College Hospital. He was an expert on gastric ulcer. It is known that he had been given special facilities for the care of gastric ulcer cases through the courtesy of his senior, Dr James Samuel Risien Russell (Registered 1887), who had been professor of Clinical Medicine at University College, but it is, perhaps, stretching matters too far to identify the latter with the Dr Russell mentioned by Francis Dewsbury [W94, 1(16)] in his letter of 22 December 1919 to Hardy regarding the payment of Ramanujan's accounts. There are numerous Russells listed in the *Medical Register* and, in the absence of further information, it is hard to make an exact identification.

This does not exhaust the list of medical men who were consulted by Ramanujan during his stay in England. Hardy's letter to J J Thomson (reproduced in §4 below) reveals that Ramanujan was also seen by Harold Batty Shaw (1867–1936), a London specialist in consumption and chest diseases; his name does not appear in the accounts.

This seems the appropriate place to add the name of the doctor who became Ramanujan's physician after his return to India. He was Dr P S Chandrasekhar (1869–1956), a professor of Hygiene and Physiology at the Madras Medical College and the first Director of the Government Tuberculosis Hospital, Madras. He had a reputation throughout India as a specialist in tuberculosis. The various places where Ramanujan resided in India after arriving in Madras on 2 April 1919 are well documented in [1, 10, 14].

3. Ramanujan's diet

As a Brahmin, Ramanujan was a very strict vegetarian and, as a typical Madrasi, he had a strong preference for spicy foods. In war-time England food was rationed and Indian comestibles were difficult to obtain, particularly outside London. He received some food parcels from his family in India, from friends and from a London company dealing in Indian products; see, for example, [14, p. 11]. A tin-lined box containing rice powder was sent to him monthly from Madras [1, p. 48]. According to Prof. K Ananda Rau, while Ramanujan was in Cambridge he "cooked his own food and his menu consisted of cooked rice, curds, fruits, sambhar [a spicy relish] with potatoes and other vegetables" [10, p. 76]. It is reported by C D Deshmukh [10, p. 79] that, as a vegetarian, Ramanujan would not eat eggs; on the other hand, it is stated in A S Ram's letter (see below) to Hardy that, while at Matlock, he had scrambled eggs for breakfast. When resident in Trinity he never dined in hall, nor did he eat any food prepared by the college kitchens, since "he suspected that even vegetarian food ordered from the college kitchen would be polluted by undetectable animal fats" [10, p. 78]. It is clear

that there were times when, because of the unavailability of acceptable Indian dishes, he was reduced to a very restricted diet of bread, milk and fried rice cakes (aplams).

There is ample evidence that as a patient his recovery was retarded by his obstinacy in dietary and other matters. There is confirmation for this dating back to 1909 [10, p. 74], before he left India, throughout his time as a patient in Matlock (see below) and after his return to India in 1919 [W94, 1(16–17)].

I reproduce below three letters, written during the Matlock period, which give a great deal of information on Ramanujan's diet and on other related points of interest. The first letter [W94, 1(6)] has been reproduced in facsimile in [12, p.87]. It is addressed to Ramanujan's friend Alampadi Subbaraya Ramalingam (1891–1953), who was the writer of the following two letters to Hardy [W94, 1(7–9)] and Ramanujan [W94, 1(10–12)]. This last letter was sent by Ram (as he was called) to Hardy for perusal and forwarding to Ramanujan; the fact that the letter has survived suggests that Hardy did not send it on. He may have felt that it was too strongly expressed. It will be clear from these letters that Ram was a very good friend to Ramanujan, going to a great deal of trouble on his behalf. He gives some information about himself in his letter to Hardy and it has proved possible to supplement this from other sources.

A S Ram graduated B.Sc. in engineering at King's College, University of London, in 1914, having registered there as a student in 1910. Although his curriculum consisted in the main of scientific subjects, it is of interest to note, in view of his list of topics discussed with Ramanujan, that he excelled as a student of divinity. He joined the Indian Service of Engineers as an assistant engineer on 6 February 1919 and, when he retired from the service in March 1944, had the status of an Executive Engineer. It appears that he returned to India before Ramanujan did. The fact that there is no record of any meeting in India between Ramanujan and A S Ram has led to the erroneous conjecture that Ram perished on the sea voyage back to India; the real reason for this lack of contact would appear to be the fact that, at the time, Ram was serving far away from Madras in the Public Works Department in Bengal.

In the letters I have inserted explanatory notes in square brackets, where this can briefly be done. In the first letter, the Tamil words for the various foodstuffs mentioned have been transliterated and appear in italics. In the last letter Ram has provided his own transliterations followed in parentheses by the Tamil originals; the latter have been omitted. Ram's abbreviations have, in most cases, been spelled out in full. The complete text of each letter is given.

Matlock House
Matlock
19th June 1918

My dear Ramalingam,

The whole of last night I had fever and my temperature this morning was about 102°. The old cook has left this place the day before yesterday. The present cook spoiled yesterday all the *appalappu* [small poppadums, called aplams by Ram, made from rice flour and fried in oil or ghee] by scorching some of them and leaving some raw (without stirring them perhaps). The curried rice was just like *akshata* [raw rice smeared with turmeric, used on ritual occasions, and not for consumption] and as hard as uncooked rice. Yesterday I had no dinner. At least I had some breakfast and plain boiled rice before. Even these she can't prepare properly now. And so please don't send me anything except *avarakkai* [broad beans], *vendakkai* [okra, also known as lady's fingers] and *nilakkadalai urundai* [groundnut balls made from roasted

groundnut and jaggery or molasses]. Don't send anything to the matron with instructions or any such thing.

Did you meet Dr Ram yesterday? I redirected a letter to you yesterday. Perhaps it may be a reply from Mr Field.

Ever yours
S. Ramanujan

(Don't send me oil, curry powder, *appalappu*. You may give me all these things after I leave this sanatorium.)

23rd June 1918

From: A S Ram,
25 Croft Terrace, Jarrow on Tyne.

To: Prof. Hardy, F. R. S.,
Cambridge.

Dear Sir,

I am very much obliged to you for your kind reply of 26th ult, informing me about the whereabouts and bad or illhealth of my friend Mr S Ramanujan.

By appointment I visited Ramanujan last Sunday [16 June] and stayed with him at the sanatorium itself till Tuesday afternoon. I must say I was shocked and horrified to find him in the thin, weak and emaciated state I found him in.

I desire to note a few observations on his state, health, food, mind etc., and also venture to make a few suggestions to you, so as to improve his health. Probably before I do these, I had better state a word or two as to the "Who is Who" of myself.

I am a native of Cuddalore, S. of Madras. In 1910 I came to England, as a student of Engineering Science. I graduated at London (King's Coll) and was serving my time as an assistant Engineer on the North Staffordshire Rly at Stoke on Trent. About April 1914, when I was in London, at 21, Cromwell Road, on a short visit, I met Ramanujan on his arrival there fresh from Madras. I bid him welcome and made him as at home as possible in a country then strange to him.

I regret I had not seen him, since then, till last week. While on the N S Railway at Stoke in September 1914, I joined H. M.'s Forces. After 16 months service in the Army, a small part of which was spent amongst the Indian Contingent, I was "temporarily released" to work "on munitions" at Messrs Palmers Shipbuilding and Iron Co. Ltd Jarrow, where I still continue to be employed as an engineer and an undischarged soldier. Lately I had a bit of luck and was one of those selected for service (civilian) in the Indian Public Works Department. Conditioned by my passing a test in Horse-riding, which will be held at Woolwich on 31st July, I have been promised, by the India Office, my release from the Army. I believe I am expected to sail for India before or about the end of September next. I am (have to) still sticking to Palmers and also manage to have some riding practice at Newcastle on Tyne. No doubt I am looking forward to my total discharge papers from my unit; before the receipt of these, I dare not leave Palmers, except with Palmers consent perhaps.

I may add, that I have been all these years, in touch with the Indian students' Departments of the India Office and also 21 Cromwell Road.

Before I resume re Ramanujan, I desire to add, that in view of rationing scheme in England, I had by last December arranged that my people should be sending me, fairly regularly, parcels of eatables for my use. I also must state that I am a vegetarian, but took to eggs in England. I may however say that apart from my not being a vegetarian by caste—Ramanujan being a Brahmin is a very strict vegetarian—I have not been, both by circumstances and otherwise, not been a vegetarian staunch and strict. My people, I mean, the members of my family are not all meateaters.

Anyway, my good people seem to have got somewhat alarmed at my going short of food and have been for the last nearly 2 months piling parcels upon parcels on me at Jarrow, so much so that it is still a problem with me as to how I am going to dispose of these; I say *still*,

because, in view of my leaving England shortly, I sent a cable home on 23rd May "Stop, sending, food, Ram".

Both to enquire of Ramanujan's health and also to see if he would care to share some of these eatables with me, I wrote to him about the middle of May, to the care of your address. Having not had a reply, I had to enquire of you.

As soon as I received your kind reply, I wrote to Ramanujan and pressed for an answer. After all I had a reply—a meagre one though—asking me to send some ghee—nearest Indian equivalent to butter—and a few Madras made pickles. Straight-away I forwarded to Matlock, two full bottles of ghee and one of gingelly oil for frying purposes. I might say, that 2 parcels containing the above 3 bottles, were simply forwarded to him *intact* as sealed by my people at Madras. You are no doubt aware of the difficulty and bother involved in packing a bottle of oil for carriage by post. Therefore I simply readdressed these 2 parcels to Ramanujan to Matlock. In the same week, I sent to him in a pigeonhole box, sample quantities of 3 different pickles or chutneys and of various other Indian preparations, and also cashew nuts or really kernels.

In indenting for and acknowledging these, I was able to elicit only 2 letters and a p.c. from him. Owing to changes in my work at Palmers, I found myself unable to postpone my holidays (annual privilege) to 31st July to suit my riding examination, and without choice I had to take the last fortnight. This holiday came rather unexpectedly. I did not feel like taking the holiday so early. I tried to do something particular during this fortnight but I was disappointed and couldn't get a start. Hence on Friday morning (14th inst) I sent a reply paid wire to Ramanujan as follows "*Could I call Sunday morning. Write immediately stating needs, Ram*". I received a reply "*certainly with pickles*". On Saturday I looked in vain for a letter from Ramanujan.

Soon after I finished my riding practice on Saturday 15th inst., I set off by the night train to Matlock, where after a cold weary journey, I arrived at about 8 a.m.

There were fortunately a few vacancies there and the doctor had kindly permitted me to stay in one of the rooms of the sanatorium. The Matron was kind enough to let me have tea with Ramanujan in his room and I had the other meals in the dining hall.

I was conversing with Ramanujan all sorts of topics, personal, political, war, Indian social, Christian missions, etc etc etc. I had not the slightest reason to suspect or think that he was "going off" at all, as one or two were presumptuous enough to think.

I also was careful to observe his temperature whenever the nurse measured it and also to see what food he was getting and how he was relishing the food.

Well, it is with regard to food that I have to write somewhat harshly and tersely and at a good length. For breakfast he was having scrambled eggs on toast and tea. Well this could hardly be improved perhaps, though I can make something better to suit Ramanujan's palate, but still for his health's sake, no improvement is desirable in this meal.

For lunch he was having plain boiled rice, chillies and mustard fried in butter, lately began having cucumber and lemon fruit. The doctor was kind enough to provide just for a day or two, some green peas specially for him. I maintain this meal could very considerably be improved.

Tea: More or less a repetition of breakfast with green onions substituted for eggs at times.

Dinner. More or less a repetition of lunch + a glass of milk.
I believe a glass of milk is given about lunch time as well.

On Monday I consulted the doctor if he recommended or prohibited any particular items of diet *e.g.* whether he objected to pungent stuff like curry. The doctor advised that Ramanujan could take anything he liked best.

Ramanujan told me on my enquiry, that he did not like porridge; not did he care for bread and milk, on which diet he had been formerly, for some months, he told me.

Naturally therefore I asked him why he did not think of going to a vegetarian sanatorium. I was told, the India Office was unable to recommend to him any vegetarian sanatorium. The

India students Department of the India Office, with its mighty influence and with knowledge renowned all over India, not to be able to suit Ramanujan, was an unwelcome surprise to me.

Ramanujan has wrong notions of what is meant by Sanatorium and Nursing Home.

Besides Ramanujan seems to have been somewhat cowed down by Dr Ram, who seems to have told him "As long as you are a patient and not well you are not free and the doctor has control over your movements". Besides I do not know how Ramanujan got into his head the impression that Dr Ram had told him or given him to understand that Ramanujan had by some agreement to stay in Matlock sanatorium at least 12 months. Anyway before I could think of shifting Ramanujan I was advised by Ramanujan to consult both Dr Ram and Prof. Hardy (*i.e.* yourself).

At once the desire to consult Dr Ram, myself, sprang in me and it was strengthened by one or two incidents.

I called on the matron and consulted her if I myself could cook something for Ramanujan. "No you can't go to the kitchen" was the answer. Then I offered to write out a recipe and asked them if they would cook to the recipe. Then there was some trouble about the old cook leaving. I was rather keen on having my stuff and recipe tried while I was there, so that I could see it was properly cooked and also if Ramanujan liked it. Well the opportunity was really denied to me till Tuesday 1 p.m.

On Monday there was prepared some mixture of macaroni custard and something else for pudding. Ramanujan did not get any pudding and on enquiry I found—this was before I went to lunch—that it was macaroni and something else. I at once requested that Ramanujan should be served with whatever pudding that is going in the dining room and it must be left to Ramanujan to eat it or leave it. Ramanujan afterwards told me that he liked macaroni prepared with cheese. It seems, unfortunately, that the sanatorium people couldn't get cheese. Well goodness I thought that the sanatorium had a priority claim for such foodstuffs. Anyway I had some last week at Jarrow and so also this week. As a matter of fact, I managed to spare some for Ramanujan and sent on to him about one lb of cheese on Friday afternoon (21st inst.).

Also, I understood from Ramanujan that the Matlock Sanatorium people couldn't get bananas for him, whereas on Tuesday afternoon, I saw a fellow traveller from Sheffield who was in the panel or ward of Dr Ram, carrying a bunch of about a dozen bananas. On enquiry I gathered they cost 4d each. Still they are obtainable.

Nextly Ramanujan complained that he used to get once *e.g.* when Dr Ram visited him, delicious rice pudding, sago pudding, etc and that he was not getting them lately. Also I understood from Ramanujan that the sanatorium people couldn't afford butter to fry his potatoes in.

Therefore you see, Prof. Hardy, that there is plenty of scope for improving his diet and I saw little chance of improving it so long as he was in that sanatorium. Also Ramanujan is pretty keen on moving to somewhere near London but feels helpless to carry out his desire. As a matter of fact Ramanujan desires to stay in London but I strongly advised him not to entertain that idea at all. His object is the easiness with which he could obtain Indian dishes and Indian condiments. I certainly do not agree with him.

However I am inclined to think that it will be better if he could be shifted to a vegetarian sanatorium or nursing home, down South or S.W. or at least to a private place under the supervision of a doctor where he could get more individual attention with regard to cooking.

Therefore I made up my mind, in consultation with Ramanujan, to break my return journey to Newcastle at Swinton, so as to enable me to visit Dr Ram and discuss the question fully with him. I rang him up on the phone and asked for an appointment for Tuesday. Dr Ram was very kind and told me almost anytime would suit him. I left Ramanujan directly after lunch on Tuesday and with a bit of luck I happened to catch the train which I got in at Belper and did not change at all till I got down to Swinton where I got out.

I was glad to see Dr Ram and his family at Mexborough. I first made myself sure that Dr

Ram had made no agreement whatever as to the length of time he should stay at Matlock with Dr Kincaird. Dr Ram also gave me some hope of Ramanujan's life and added that food alone could save him. I quite believe it.

Dr Ram asserts that there are very few doctors who would care to have Ramanujan in their nursing homes and fewer still who would bother with humouring Ramanujan's palate. It is only after I saw Dr Ram that I was somewhat impressed with the nonpliability and the inadaptability of Ramanujan's palate. Also Dr Ram placed before me the enormous difficulties that are involved in running a separate say 4 roomed cottage with perhaps a special nurse and a special cook and probably a servant in addition.

In any case when I left Dr Ram on Tuesday at about 9 p.m. I was better acquainted with the difficulties of Ramanujan's case; also that Ramanujan had too particular a palate to be easily pleased.

Now then I will turn to the solution of the problem. Though Dr Ram thinks that Dr Kincaird is the only kind and charitable doctor that will tolerate Ramanujan, I am hopeful—I have consulted a few of my friends and doctors out here—that I could find one or two nursing homes which will suit Ramanujan better. Though my desire is to move him to near the sunny S. or S.W. the few homes that I got the addresses of, so far are all situated in S.E. or near London. I desire to avoid these because of the air raids. However I will try and place my results before you shortly.

This question appeals to me in 3 main items

(a) The desirability of an operation by a skilled surgeon and specialist like Dr Boyd of Newcastle, thorax specialist. The operation should not be undertaken just for practice or for small chances. The purpose and location should be clearly defined beforehand. If there is an ulcer, well if it is not too far gone, there is a chance of its being removed by operation; if there is no ulcer, the idea of operation should be thought out beforehand not only more than once but very soundly and several times.

(b) Merely to have the same treatment *viz.* fresh air treatment—as he is having now. In this case, the possibility of moving him to a place, preferably vegetarian, where he will be better looked after.

(c) Both of the above are conditioned by Ramanujan's being in a state fit to be transported and no doubt of the availability of transport.

Well I do not know if (a) is decided upon, it can be carried out at Matlock. It means the specialist surgeon will have to be at or go to Matlock.

So that I could consult my friends—both doctors and others over here— I jotted down a few particulars about Ramanujan's disease and the theories as diagnosed by the four or five doctors he came under. Only what I put down on paper was Ramanujan's statement. But my friends over here would rather have a medical statement by say Dr Kincaird, describing all the particulars and also summing them up for judgment by my friends.

I however think it won't be fair if *I* asked Dr Kincaird for such a report. You pay the doctor and it is reasonable that you have a far better right to ask for such a report, than myself. Therefore if you would rather that I continued my efforts, I should be obliged for placing such a report in my hands at your earliest convenience.

I need hardly add, that I will treat the report as confidential and will not use it in any way objectionable.

Well as a matter of fact I desired to consult Dr Kincaird privately *i.e.* outside the reach of Ramanujan's hearing—but Dr Kincaird was so busy, I believe, that he forgot to fix me up with 5 minutes.

Still a soldier as I am, I am confident of getting away to visit Ramanujan or help Ramanujan with transport. I do not know if it is anything of value but I desire to place it before you for what it is worth. I have passed the second year's examination in ambulance or first aid work

and for Ramanujan's sake, I do not mind qualifying in sick nursing.

When you send for Dr Kincaird's medical statement I do not know if it is advisable that you should ask to be enlightened about this point. I personally think it is desirable that some light should be thrown on this procedure. It is this. Dr Kincaird *distinctly* told me that Ramanujan could eat anything that he liked and that nothing was injurious or objectionable.

On the other hand I hear from my friends in Jarrow that eatables like pickles, chillies etc should be avoided by consumptive patients. There is one reason however in which case these can be excused. When the patient has gone too far to be remedied, and when it is a question of only a few weeks or months, it matters little, as long as the patient feels happy and comfortable in the last days of his life. In permitting me to give anything to Ramanujan to eat, is this the object of Dr Kincaird?

It is anguishing and breaking one's heart to feel that Ramanujan, with his wonderful capabilities and valuable contributions, should be given up for such as hopeless. War with all its horrors might have made us callous to the whole-sale slaughter and loss of lives but surely should Ramanujan be given up?

Now I desire to place before you certain proposals. Before I do that I desire to call your attention to this fact. Ramanujan is not only a vegetarian but is one with a strong Madras palate; what is worse he is suspicious of anything that is offered to him being meat. I will write to him and advise him that he should choose between killing—no, controlling will do—his palate and killing himself. He is thinking of his vegetarianism even at the expense of his health and life. But one cannot but think of him as cranky or headstrong when he refuses cream and lay plums on it; he also dislikes porridge or oatmeal. The solution is to provide him with Madras dishes; but is it practicable; are they commendable for his health?

Nextly might I call your attention to his letter (original) of 19th inst., since I left him. It is enclosed herewith. The stuff—call them *Aplams* if you like—I have marked blue on his letter is good for him. It is neither hot nor pungent nor acidy nor sour nor sweet. It simply wants frying in oil which I did provide. It is fried exactly like chips. As a matter of fact I was surprised to find, when I went to Matlock, that the bottles of oil and ghee had not been opened into or used at all. Ramanujan's explanation was that he did not then feel like eating these preparations or did not feel well enough to eat these. This is not true because he had eaten the dried vegetables (marked red on his letter)—call it Avarakai—raw without being cooked. The latter won't do because uncooked avarakai only throws additional work on the digestive organs.

This is why on Thursday I fried here (at Jarrow) some aplams and sent them on to him, together with some avarakai. Aplams once fried are eatable and alright cold but certainly far better warm—not warmed up.

Now then I come to 3 somewhat bold proposals impracticable perhaps but not impossible. All these three are dependent upon influence exerted by you thro' the India Office and also Ramanujan's ability as an F.R.S.

(i) The very first is the possibility of shifting Ramanujan to Southern France or Italy. Of course India will be better but that is impracticable. In case he is permitted to be moved to S. France or S. Italy, he can be moved on a Red Cross carriage or wagon and a hospital ship. The sea voyage is so comparatively short. The climate of S. Italy is a very effective remedy for Ramanujan.

(ii) In addition to (i) or even without (i), the "temporary release" of a soldier cook (Indian) to work for Ramanujan for making his Indian diet. I mention a soldier cook because a soldier cook appreciates his freedom from military discipline and is more or less faithful to his civilian job. It may be argued that this is misusing soldiers. So many soldiers on munitions there are at present like myself. It is more or less taking advantage of the fact that we are soldiers and putting us to where they want us and not necessarily to where we are fit for.

I dare say I can get hold of one or two lascars—coloured workers on mercantile marine—knocking about on the Tyneside. Amongst them are some good cooks; but

they are extremely unreliable. They will desert Ramanujan as quickly as they have deserted their ships.

(iii) Even in case of moving Ramanujan within England, it will be much better to get permission for transporting him on a Red Cross van or Rly carriage.

There is little else that I can usefully add now; but I am busy trying to get some valuable information and I will place these before you shortly.

Do you think you can still fall on your first offer from Bournemouth? I mean the one that Ramanujan had to go to, if Dr Ram had not brought him to Mexborough.

I desire to meet you at Matlock when you will call on Ramanujan at the end of this session. Therefore I left a stamped and addressed telegraph form in Ramanujan's writing box, so that he loses no time in informing me of your arrival.

In case I wire to you and you confuse my name with that of Dr Ram, please look for place of origin. Or if the telegram is from London, say, I will sign myself *Sram* to denote S Ram. Dr Ram is L Ram.

I regret to have made my letter so very long but I could hardly help it. One would think that such a question cannot be thrashed out on paper. But still I have little other choice.

I look forward to your kind advice and guidance.

Believe me, Dear Sir
I am,

Yours very sincerely,
(Sgd) A S Ram.

P.S.

I am enclosing my letter to Ramanujan herewith. Please peruse it and post it straight away. I trust you follow my position and case clearly. *ASR*

A. S. Ram, 25 Croft Terrace,
Jarrow on Tyne. Sunday 23rd June 1918.

My dear Ramanujan,

I was exceedingly grieved to have your painful letter of 19th inst. Sorry to hear the new cook is a failure so far as you are concerned. How did you like the aplams cold? I mean those that I sent on to you on Thursday last (20th inst.).

Please ask matron to boil those avarakais for you. They shouldn't find it difficult to do that. It only means another sauce-pan on the fire.

Also you told me you liked macaroni prepared with cheese. I forwarded about 1 lb—possibly less—of cheese on Friday 21st inst. Please instruct matron to have these prepared to your taste.

I am starting work tomorrow in the shop as a mechanic. I have to turn up at 6 a.m. for the first time for the last over 18 months. Well I am glad it won't be long; at the same time it is an opportunity to train my hands.

I had a reply from Mr Field—that was not the letter you so kindly and promptly forwarded. The India Office does not recommend private arrangements. Therefore my plan to go as a marine engineer falls thro'.

Possibly tomorrow I will be forwarding to you a parcel containing (a) ground nut balls (b) vegetable fingers and (c) Avarakai. These were the only 3 you have asked for in yours 19th/18.

By the bye, I am forwarding your letter 19th/18 to Prof. Hardy.

I am writing to Prof. Hardy today and making a sound case for you. I hope to have his advice soon.

Now then I will have to be a bit harsh with you.

Both from my talk with Dr Ram and after my second thoughts, I am impressed with your being so particular about your palate. Well you will have to choose between killing or controlling your palate and killing yourself. You must try and get yourself to like porridge or oatmeal, cream etc. My friends have strongly advised me not to let you indulge in pickles and chillies. You know you can train your palate. Dr Ram told a patient who expressed unwillingness to take milk

"You must take milk or you will go to Hell".

I am not going to the extremes and asking you to take beef tea or bovril, though considering your life, my asking you to take such a thing is quite excusable nay desirable and even unavoidable.

Be reasonable and don't be bigoted. Practise self restraint and not stick to curries and pickles. I am conserving all my ghee and oil for you.

Please provide me a weekly report stating among other things your normal morning and evening temperatures. State what food you are getting and what food you would like me to send.

Tinned Maize is very hard to get but I have not given up my search. There is little hope of my getting any dessicated coconut. Unfortunately coconut cakes and coconut biscuits cannot be well made with ordinary moist coconut kernel.

From the last few lines of your letter 19th/18 I gather you are confident of being moved from Matlock Sanatorium. Even if it were otherwise possible, are you in a state fit to be moved? Certainly not.

Well it is up to you therefore, to eat or drink plenty of porridge, eat tomatoes, bananas, cheese, cream, macaroni etc. Bananas and cream go together lovely. Eat these and fatten yourself and then alone, the possibility of your moving will be considered.

Dr Ram has certainly made no agreement whatever as to the length of time you should stay or will stay at Matlock House.

He gives you plenty of hope to recover but he makes it entirely dependent upon food. Have you any objection to taking foods like cod liver oil, sanatogen which aim at strength building and also fattening.

Now then hurry up and begin eating or devouring plenty and get fat. I shall take you with me to India. That is a good boy.

I am extremely busy. You are constantly in my thoughts and prayers.

Hoping to take you on my visit to London next month.

I am,

Yours

A S Ram.

P.S.

Johnny Guardplate—I mean guard against evil spirits etc,—that I left with you, doing you any good? Please keep it carefully or send it on to me when you feel inclined.

A.S.R.

I say Ramanujan, my cadet friend J B Madgshon is taking his final examination for his commission at Cambridge about 28th inst and will be going over to Newcastle soon after. I could ask him to fetch your stuff—I mean Indian eatables and grain etc—and then I could take them and present to Dr Ram if you like.

However if you want anything done in this respect you must hurry up and write to me by return. You must give full instruction as to where he could get the key etc and where exactly you have kept these.

A.S.R.

The Mr Field referred to in Ramanujan's letter and Ram's reply was Robert Edwin Field, a minor staff clerk in the department of the Secretary for Indian students at the

India Office. The Department of the Educational Adviser to Indian Students was housed in 21 Cromwell Road, London. Immediately after his arrival in London aboard the S S Nevasa, Ramanujan was taken there and it was there that he first met A S Ram.

In his letter to Hardy, Ram refers several times to Dr Kincaird; as indicated earlier, the correct spelling is Kincaid. The thoracic specialist mentioned in the letter may be John William Hugh Boyd, who graduated MB, BS from the University of Durham in 1899 and practised in Newcastle upon Tyne. There is no evidence that he was ever consulted about Ramanujan.

There is a suggestion at the end of the letter that Ramanujan might have gone at an earlier stage to Bournemouth for treatment. I have found no other reference to this possibility, nor do I know why it should have been Dr Ram who "brought him to Mexborough". L Ram was a medical graduate (in 1904) of the University of Edinburgh and practised continuously in Mexborough from 1907 onwards. He appears to have had no Cambridge connexion, but may have been recommended by one of Ramanujan's Indian friends. He was the author of a paper on abdominal dropsy and presumably, being in charge of a ward at Matlock House, was skilled in the treatment of chest diseases.

Unfortunately, Hardy's replies to A S Ram's letter have not survived, so that it is not possible to determine what action may have been taken to meet Ram's various suggestions, or whether Ram was responsible for Ramanujan's admission to Fitzroy House or Colinette Road.

4. Symptoms and diagnosis

No official medical reports on the nature of Ramanujan's illness during his time in England have survived. Perhaps the fullest account is contained in a letter (reproduced below) from Hardy to the distinguished physicist (1856–1940) Sir Joseph John Thomson, who, in addition to being President of the Royal Society at the time, became Master of Trinity College in February 1918. The letter concerns Ramanujan's candidature for a Fellowship of the Royal Society and is part of a collection of Thomson papers housed in Cambridge University Library; the reference is Add. 7654/H27.

TRINITY COLLEGE,
CAMBRIDGE.

Dear Thomson

I shd like to say more clearly what the position about Ramanujan is. I wouldn't have done so if you hadn't asked me; I am much afraid of prejudicing him by persistent pushing, in view of his double candidature for a Fellowship and the R.S.

If he had not been ill I wd have deferred putting him up a year or so: not that there is any question of the strength of his claim, but merely to let things take their ordinary course. As it is, I felt no time must be lost. I hardly hoped that he wd have much chance this year—I don't know whether he has or not. If he has no doubt his illness will have much to do with it, and I should tell you precisely how (so far as I can know) matters stand.

Batty Shaw [see §2] found out, what other doctors didn't know, that he had undergone an operation about 4 years ago. His worst theory was that this had *really* been for the removal of a malignant growth, wrongly diagnosed. In view of the fact that R. is no worse than 6 months

ago, he has now abandoned this theory—the other doctors never gave it any support.

Tubercle has been the provisionally accepted theory, apart from this, since the original idea of gastric ulcer was given up. There is a perfectly regular rise of temperature every night, and persistent weakness. But there is certainly no crude lung trouble: no coughing or spitting. And Dr Kincaid, who runs Matlock, told me he is quite doubtful still, thought it might be some obscure Oriental germ trouble imperfectly studied at present.

For my own part, I think *probably* he will be alive in a years time, and that he *may* recover completely. If he were like Jeans—a man who made a scientific fight to get right—he would have a much better chance. Like all Indians, he is fatalistic, and it is terribly hard to get him to take care of himself. Everyone, too, is frightened of the continual illness and solitude affecting his mind.

As I said, I am nervous about trying to rush him; and I am aware that for the time being I am not an ideal supporter. And I realise that the R.S. has many other things to consider. But there is no doubt that (especially after his disappointment in the Fellowships) any striking recognition now might be a tremendous thing for him. It wd make him feel that he was a success, and that it was worth while going on trying. It is this much more than the fear of the R.S. losing him entirely which seems to me important.

I write on the hypothesis that his claims are such as, in the long run in any case, could not be denied. This is to me quite obvious. There is an absolute *gulf* between him and all other mathematical candidates.

Yours sincerely
G H Hardy

P.S. I forget whether I ever sent you a copy of my report on him. In case I didn't I send another.

I also send a copy of some suggestions I made to the CPS [Cambridge Philosophical Society] Council about publications. They were referred to a Committee of the Officers. As so much of the Proc. is Physics, your opinion wd be most helpful.

Hardy's letter is undated, but would have been written before 21 February 1918, probably early in that month. For the candidates for election to the Royal Society were considered by the Council of the Society firstly on 14 February and again on 21 February, when the list was drawn up, including Ramanujan's name. This provisional list was read out at the meeting on 28 February, the names were published in *Nature* on 7 March and the election finally took place on 2 May 1918. It is of interest that James Hopwood Jeans (1877–1946), who later was knighted, and to whom Hardy makes reference, was a member of the Council at this time.

It appears that the first diagnosis of gastric ulcer was later changed, on not very firm evidence, to tuberculosis. By the end of 1918 his health had improved, and his election to the Royal Society and a Fellowship at Trinity may have been instrumental in this. In his letter [1, p. 75; 14, p.76] of 26 November 1918 to Francis Dewsbury, Hardy states:

> There is at last, I am profoundly glad to say, a quite definite change for the better. I think we may now hope that he has turned the corner, and is on the road to a real recovery. His temperature has ceased to be irregular, and he has gained nearly a stone in weight. The consensus of medical opinion is that he has been suffering from some obscure source of blood poisoning, which has now dried up: and that it is reasonable to expect him to recover his health completely and if all goes well fairly rapidly.

Evidently, by this time the diagnosis of tuberculosis had been abandoned by his English doctors. On his return to India his health did not improve further, as had been

hoped. Tuberculosis was again the preferred diagnosis. I am indebted to Prof. S Chandrasekhar, F.R.S. of the Enrico Fermi Institute of the University of Chicago for sending me the following extract of a letter written to him by A Ranganathan the grandson of Dr P S Chandrasekhar, Ramanujan's physician (see §2). Mr Ranganathan quotes a passage from his grandfather's diary, written on 27 April 1920 (the day after Ramanujan's death):

> Ramanujan's death—on April 26, 1920—could have been avoided. If he had been allowed to follow my instructions, this double tragedy need not have taken place. The neglect of Ramanujan during his early phase—perhaps partly due to the ignorance of his contemporaries—. . . contributed to this double tragedy—a tragedy which is too deep for tears. I cannot get over a particular experience—in fact it was an unusual experience for a physician. Tragically enough, R himself told me that he had lost the will to live on several occasions. Again R told me that he shouldn't have returned to India. His death is the saddest event of my professional career. It is not for me to assess R's mathematical genius. But at the human level, he was one of the noblest men I have met in my life—shy, reserved and endowed with an infinite capacity to bear the agonies of the mind and spirit with fortitude.

Ramanujan's earlier surgical operation is mentioned in the third paragraph of Hardy's letter to J J Thomson. This was for the removal of a hydrocele and occurred in 1909, much earlier than 4 years previously; see [1, p. 106], [14, p. 178] and [7, p. 110].

We also learn from Hardy's letter what has not been publicly revealed before, namely that Ramanujan had been unsuccessful in an earlier application for a Fellowship at Trinity. His successful election came later in that year on 13 October 1918.

There are various references in correspondence to Ramanujan's state of mind during the latter part of his period of residence in England: in Hardy's letter to Thomson, in A S Ram's letter of 23 June 1918 to Hardy, and in Hardy's letter of 26 November 1918 to Dewsbury, already referred to, where he writes "At various times we have felt considerable anxiety about his mental state. I do not think there is really anything seriously amiss with it. But the long illness, and spells of comparative solitude have undeniably had an effect and he has been subject to fits of depression and been difficult to manage". These passages may refer to his attempted suicide by falling in front of a train in an underground tube station in London; fortunately the train stopped short in time. The exact date of this incident is unknown, but is more likely to have been February 1918 (as Prof. S. Chandrasekhar informs me) than the second half of 1917, as claimed by Nandy [7, pp. 126–127]. For it occurred before Ramanujan had been officially elected an F.R.S., but at a time when it was known that this was likely to occur.

It is probable that Ramanujan's fits of depression were the result of the rather solitary life he lived while in Cambridge and in the various nursing homes where he was a patient. For this his working habits and vegetarianism were largely to blame. B M Wilson [W107C, 11 (1–6)], who was an undergraduate in Trinity at the time, notes that Ramanujan was very rarely seen in Cambridge and that he sometimes worked for 30 or so hours at a stretch and then slept for 20. In a Cambridge college an important venue for social contacts is the dining hall, and this Ramanujan never entered, as he did all his own cooking. Moreover, a large number of the Cambridge mathematical dons were away on war service. Neville was no longer available; his Fellowship at Trinity terminated in 1917 and he moved to London, before taking up the Chair of

Mathematics at Reading in 1919. On the other hand, Hardy [3, p. 11] saw Ramanujan almost every day. He did, of course, have several Indian friends and liked to visit London with them during vacations and rent a flat there [9, p. 45].

After the lapse of more than sixty years we cannot hope to determine beyond doubt the exact nature and cause of Ramanujan's illness, and can only conclude that medical knowledge at the time was not sufficiently advanced. The study of vitamins was still in its infancy, and it could be, as has been suggested to me by Dr Richard Askey of the University of Wisconsin-Madison, that, during the latter part of his stay in England, Ramanujan was suffering from a vitamin B_{12} deficiency caused by his very restricted diet and aggravated by cold climatic conditions. A distinguished medical colleague of mine confirms that this is a possibility and points out that this deficiency could have predisposed him to infections.

5. Concluding remarks

For convenience I divide these into a number of subsections.

5.1 *Hardy's precursors*

The debt owed by the mathematical community and by Ramanujan to Hardy cannot be exaggerated. He responded generously to Ramanujan's letter of 16 January 1913 and, in the following seven years, went out of his way to befriend and encourage him. Having an outstanding reputation as an analyst, Hardy was an obvious person for Ramanujan to approach, but, as is well known, he was not the first mathematician outside India to be asked to assess his work. In fact, it appears that at least *three* other mathematicians were written to, although it is not certain (as Prof. K G Ramanathan has suggested to me) that they all received the full statement of work that was sent to Hardy.

In [12] I have given some information about Prof. M G M Hill of University College, London, who was approached through his former pupil C L T Griffith, who was Professor of Civil Engineering at Madras College of Engineering. At the time of writing I had not seen the letter of 3 December 1912 (reproduced in [12, pp. 53–54]) written by Hill to Griffith making some detailed comments on Ramanujan's work; this was more helpful than my previous information indicated, if not exactly encouraging.

Following a conversation with Littlewood, Nandy [6, pp. 115, 146–147] states: "Hardy was the third person [but see above] to receive samples of Ramanujan's work; two English mathematicians had earlier returned them without comment. Littlewood says with some relish that these two mathematicians, whom he identifies only as Baker and Popson, felt rather foolish afterwards." The first named is Henry Frederick Baker (1866–1956) and the second is, doubtless, Earnest William Hobson (1856–1933). Both were distinguished analysts, although the former is now better known for his later work on the foundations of projective geometry. Baker was Hardy's predecessor as Cayley Lecturer at Cambridge and was appointed Lowndean Professor of Astronomy and Geometry in 1914. Hobson was appointed Sadlerian Professor of Mathematics at Cambridge in 1910. It has been said that he first (with W H Young) brought the modern theory of real functions to Cambridge and Britain.

Professional mathematicians are accustomed to receiving unsolicited material from amateurs and cranks and know, from bitter experience, that to reply can be abortive

and time-consuming. We may regret that Ramanujan's genius was not at this stage immediately and universally recognized, but, as C P Snow has stated in his foreword to Hardy's *A mathematician's apology* [4, p. 34], we cannot help having a sneaking sympathy for those who returned the material to Ramanujan without comment.

5.2 *Eric Harold Neville*

In [12, p. 81], it was stated that Ramanujan sailed to England, on 17 March 1914, in the company of E H Neville. It was the original intention that they should travel together, but this proved impossible to arrange. According to [1, pp. 43, 71, 109], Neville left three days earlier on 14 March 1914; however, Neville's own account [8, p. 294] would seem to indicate a somewhat earlier departure. He met Ramanujan on his arrival on 14 April in England at the docks. According to information received by me from his younger brother Raymond, Eric Neville was accompanied by his older brother, who possessed a Jowett car in which all three drove to Cromwell Road. Four days later Ramanujan arrived in Cambridge [14, p. 5] and stayed with Neville and his wife in their house at 113 Chesterton Road; this tall three-storey house is still standing. From there he moved to rooms in Trinity College in July 1914.

Neville's article [8] was originally prepared for a talk broadcast in Hindustani in the Indian radio service on 22 April 1941. It is based on a fuller version now deposited in Box 120 at the University of Reading Record Centre. From the material omitted from the published article I select two items of interest. The first concerns Ramanujan's fluency in English:

> If, as we are told, imperfect knowledge of the English language was an element in his failure at Kumbakonam, ten years had worked wonders, for a more fluent speaker or one with a wider and better used vocabulary I have seldom met.

This refers to Neville's visit to Madras in 1914. As an example of Ramanujan's felicitous choice of words one may instance his introduction of the term 'mock theta-function'.

The second item illustrates the social disadvantages then suffered by conservative Indians on their return home, as a consequence of the 'pollution' incurred by crossing the seas:

> "When I go back", he said to me once with a sigh, "I shall never be asked to a funeral".

5.3 *Ramanujan's notebooks*

Professor Bruce Berndt points out to me that the statement on p. 82 of [12] that Notebook 2 is an amplified and improved version of Notebook 1 is somewhat misleading in that the second notebook differs from the first almost from the beginning. He also notes that B M Wilson took responsibility for Chapters I–XII (not I–XIII as stated on p. 86 of [12]) of Notebook 2.

In [10, pp. 56–57] Ranganathan recounts that in March 1925 Hardy handed over to him the original of Notebook 1, which Ramanujan had left with him in Cambridge on his return to India, and states that later the University of Madras gave funds for all three Notebooks to be transcribed and that one of the copies was sent to Hardy; from the context it appears that this was a copy of Notebook 1 only. There is some slight

discrepancy here, since in earlier letters written by Francis Dewsbury on 2 and August 1923 [W94, 12(25, 27)] Hardy is requested to return Notebook 1 for sa keeping to the Madras University Library and in a subsequent letter of 30 August 19 [W94, 12(28)] Dewsbury informs Hardy that he has dispatched to him a transcript Notebook 2 and of the three quarterly reports made by Ramanujan. It is likely that th transcript [W101–104], which is in four manuscript volumes, also contained the mu shorter Notebook 3; Watson in his description of the notebooks [15] makes r reference to this third notebook. In Ranganathan's account of his conversation wi Hardy in March 1925 there is no mention of the copy of Notebook 2, whic presumably, Hardy had at that time, unless he had lent it to B M Wilson.

5.4 *Other letters and papers*

The comments in this subsection refer to my earlier paper [12]. In it I omitted to sta that the collection of letters [W94, 1(1–32)] was sent by Littlewood to Watson in Ju 1948, after Hardy's death. These letters contain no mathematics and include the thr letters reproduced in §3.

On p. 93 I conjectured that, after B M Wilson's death in 1935, his Ramanujan pape were dispatched by Dr Frederick Bath (1900–1982) to Watson or Hardy. I can no confirm that they were sent to Watson, since the contents of [W107C, 1] were includ in an envelope (now very tattered) bearing on its outside the note:

> To Prof. G N Watson, Birmingham
>
> MSS Notes of BMW on Ramanujan! Note Books covering several chapters. Also some M of BMW's own papers as far as I can ascertain. You will be able to decide whether these ha any bearing on R's work and destroy them if they are useless.
>
> F.

On p. 94 it was stated that Ramanujan's letter of 28 June 1918 to Hardy [W94, 2(19 was written from Matlock. It is, in fact, the only one of the group of four letters referr to that does not bear an address, and may have been written from London. At the en of the letter he remarks that he is sending a cheque for £100; this is, no doubt, the su appearing in the accounts (see §2) as received on 4 July or 22 October 1918.

Ramanujan's last letter to Hardy [W94, 14(1–3)], written on 12 January 1920, is n longer fully extant. Only the mathematical parts survive and they were reproduced [5]; see p. 86 of my paper. What he wrote concerning his health, we do not kno although the letter evidently made reference to his debts and subscriptions mathematical societies. This is known from Dewsbury's letter of 15 January 1920 Hardy [W94, 1(17)].

I note, finally, that Sir Edward Wright informs me that he was the pupil of Hardy mentioned in the postscript to the letter to G N Watson quoted on p. 85.

5.5 *Hardy's social relationships*

In his psychological study of Ramanujan [7] Nandy makes a number of assertio about Hardy and his relationships with other people. Although I have no psych logical qualifications, I feel I must contest some of these statements, if only to ensu that future writers do not assume them to be generally accepted. I select one passa [7, p. 117] for especial comment:

> A son of a middle class family of teachers, Hardy always felt an outsider within the British elite culture of which Oxbridge was the ultimate academic symbol. Contemporaries recall his particularity for non-white students and colleagues, and his shyness and withdrawal in dealing with the progenies of the English gentlemanly class who crowded these prestige universities.

I believe this to be an erroneous assessment. To establish my standing in this discussion, I mention that I was one of Hardy's research students. It was during the years 1938–1940 that I had the most frequent contact with him—more than twice a week during term time—, speaking to him after lectures, during his weekly 'conversation class', or in his rooms. Moreover, I have consulted more senior colleagues whose knowledge of him goes back much further.

Hardy was a shy man in the sense that he would rather say nothing than indulge in small talk or the trivia of polite conversation. In mixed company he tended to ignore members of the other sex, unless they happened to be mathematicians. He had a detestation for ceremonies and formal occasions of all kinds, not only religious ones, and, for example, rarely attended college feasts. This dislike extended even to such mundane matters as shaking hands on first acquaintance. I remember his telling me that, when he was walking in Copenhagen with his friend Harald Bohr, he kept his hands firmly in his pockets to avoid shaking hands with the numerous people introduced to him by Bohr.

These remarks may give the impression that he was conceited and stand-offish. This was not so. He was very kind to his students (of whatever race or colour), concerned for their interests and never talked down to them. As Ramanujan wrote in his letter to Krishna Rao on 11 June 1914 [14, p. 7]: "Mr Hardy, Mr Neville and others here are very unassuming, kind and obliging."

To claim that he "felt an outsider within the British elite culture" is nonsense. He had been at Winchester College as a boy and no one who had been at that famous Public School would have had such feelings, even if he had found the spartan regime there disagreeable. As a young Fellow of Trinity he was an active member, and later secretary, of the very select and semi-secret (although nowadays somewhat notorious) Apostles society, an 'intellectual aristocracy' whose members included such figures as G E Moore, J M E McTaggart, G M Trevelyan and Bertrand (later Earl) Russell. It is relevant to quote from C P Snow's foreword [4, p. 42]:

> He had been born . . . into a professional family: almost all his life was spent among the *haute bourgeoisie*: but in fact he behaved much more like an aristocrat, or more exactly like one of the romantic projections of an aristocrat . . . Underneath his shyness, he just didn't give a damn.

The "progenies of the English gentlemanly class" who "crowd" Oxford and Cambridge rarely crowd mathematical lecture rooms or senior common rooms, so that Hardy's contacts with them must have been minimal and, in any case, insufficient to base any firm conclusions as regards his behaviour and feelings in their company.

5.6 *Ramanujan as a mathematician*

Although I have been concerned in this article primarily with Ramanujan as a patient, it might seem peculiar to conclude without making any comment on him as a mathematician. Accordingly I take this opportunity to give a personal view of his stature and reputation as a mathematician in the 1980s.

It is now over 60 years since he died and it is, probably, still too soon to make a final assessment. In [5, p. xxxiv] Hardy gave a brief estimate of Ramanujan's "finest, most independent, and most characteristic work" based on his published papers, and I do not think that any mathematician familiar with his work would wish to omit any of the subjects selected at that time. Over the intervening years, however, our knowledge of his unpublished work has greatly increased, mainly due to the labours of Bruce C Berndt and George E Andrews, and it is now possible to add to Hardy's list a number of topics, where Ramanujan displayed his remarkable inventive talents or showed a deep insight, and where he has stimulated other mathematicians to obtain results of importance.

I give four examples, but before doing so, would remark that I do not think there is any evidence for the suggestion [7, p. 149] that his work on fractional differentiation came closest to being a major breakthrough in mathematics. This statement may be based on the account of the subject given by Ranganathan [10, pp. 106–107], who, however, makes no such claim. Fractional differentiation has a long history. Leibnitz commented on its possibility in 1695 and the subject was extensively discussed in the first half of the nineteenth century, by Lacroix (1819), Abel (1823) and more fully by Liouville (1832); see [13] for a more detailed account.

My first two examples concern congruences satisfied by certain arithmetical functions, namely the Fourier coefficients of certain modular forms, although Ramanujan did not use the latter terminology. Such congruences are of two kinds. The first kind states that the function considered, for example Ramanujan's function $\tau(n)$, satisfies a certain congruence, usually containing divisor functions, for all positive integers n, or for all n in certain residue classes. A few such congruences were published by Ramanujan, but his unpublished manuscripts contain many more, including several that were later rediscovered by other mathematicians.

By 1940 a considerable number of congruences of this type had been found and (*pace* André Weil [16]) many number-theorists agreed (and in my view, justifiably) with Hardy [3, p. 161] in wondering whether the writers of these papers were "straying into one of the backwaters of mathematics"; for the subject seemed a curious rag-bag of uncoordinated, if interesting, results. This view has now been overturned as a result of the work of J P Serre and H P F Swinnerton-Dyer who, by using l-adic representation theory, have placed the whole subject in a general setting which sheds light on how such results arise.

The second kind of congruence states that a given arithmetical function $a(n)$, say, is divisible by a fixed positive integer l for almost all n, and an asymptotic formula or upper bound may be obtained for the sum of those integers $n \leqslant x$ for which this does not hold. In the unpublished parts of his manuscript [W94, 13] Ramanujan obtained several results of this nature and, in view of his usual avoidance of use of any complex function theory, it is of interest that he refers to the use of what he calls "transcendental methods" to obtain his asymptotic formulae. These results hold, in the first instance, for the moduli l that occur in congruences of the first kind, but he gives evidence (see [11, §3]) of his remarkably deep insight by considering the problem in general and suggesting that the results extend to other moduli (*e.g.* 11 in the case of his function $\tau(n)$). This he is unable to prove, but it is noteworthy that his general argument has some similarities with the methods used recently by Serre, who has successfully studied the more general problem.

My third example comes from what Birch [2, §3] has called Ramanujan's Fragment

V. This is a list of assertions that the Dirichlet series associated with various modular forms have Euler products. I doubt whether Ramanujan ever consciously used the concept of linear independence, but what is remarkable here is that he realized that (to express his results in modern terms), for modular forms of higher weight, where the associated vector space has dimension greater than 1, one could find a number of linearly independent modular forms each having an associated Euler product. He gives many examples involving quadratic surds for the coefficients of the basic forms. This is work that had to wait until 1937 before first being put on a firm foundation by Hecke.

My last example concerns the deep work done by Ramanujan on mock theta functions and related q-series during the last year of his life, after he had returned to India. This is contained in a collection of 87 loose unordered sheets of paper [W94, 15(1–87)], and we are indebted to George E Andrews, who first recognized its importance and has assumed the difficult task of examining the material in detail and publishing proofs.

Further examples could probably be given, but I think I have given enough evidence to justify my contention that Ramanujan's reputation as a mathematician, although very high at the time of his death, has increased considerably since that time.

Undoubtedly, his most fruitful period was the six years between his arrival in Cambridge and death in Madras. What he accomplished during this time is truly remarkable, particularly when one considers that he was rarely out of bed during his last three years.

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Mean-value of the Riemann zeta-function on the critical line

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Abstract. This is an expository article. It is a collection of some important results on the mean-value of $|\zeta(\frac{1}{2}+it)|$.

Keywords. Mean value; zeta-function; critical line.

1. Introduction

This article is a collection of some important results on the mean value of $|\zeta(\frac{1}{2}+it)|$ and its powers. Section 2 deals with the mean square and §3 with the mean fourth power and the mean twelfth power. Section 4 deals with the general $(2k)$th power where $k > 0$ and §5 gives other important results.

2. Mean square

In 1918 Hardy and Littlewood [12] proved that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \, dt \sim T \log T.$$

In 1926 this was improved by Ingham [18] to

$$\frac{1}{2\pi}\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \, dt = \frac{T}{2\pi}\log\frac{T}{2\pi} + (2\gamma - 1)\frac{T}{2\pi} + E(T),$$

where $E(T) = 0\ (T^{1/2}\log T)$. More complicated argument due to Titchmarsh [33] showed that $E(T) = 0\,(T^{5/12}(\log T)^2)$. The final improvement of this theorem was published in 1978 by Balasubramanian [3] who proved that $E(T) = 0(T^{1/3})$, $E(T) = 0(T^{27/82})$, and so on. It should be mentioned that subsequent to this discovery Good [10] gave a variant of Balasubramanian's proof. It should also be mentioned that Heath-Brown's powerful method [14] which gives the mean fourth power with error term also gives the theorem of Balasubramanian.

Titchmarsh [34, 35] had introduced a new mean value. He had proved that, as $\delta \to 0$

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 \exp(-\delta t)\, dt \sim \frac{1}{\delta}\log\frac{1}{\delta}$$

and

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 \exp(-\delta t)\, dt \sim \frac{1}{2\pi^2\delta}\left(\log\frac{1}{\delta}\right)^4.$$

Kober [22] proved that

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 \exp(-2\delta t)\,dt = \frac{\gamma - \log(4\pi\delta)}{2\sin\delta} + \sum_{n=0}^{N} c_n \delta^n + 0(\delta^{N+1}),$$

where the 0-constant depends on N. A simpler proof was given by Atkinson [1]. Regarding Ω theorems for $E(T)$ Good [11] developed a method to prove that $E(T) = \Omega(T^{1/4})$. Heath-Brown [15] developed a more powerful method which actually gives

$$\int_0^T (E(t))^2\,dt = cT^{3/2} + 0(T^{5/4}(\log T)^2),$$

where $c = \frac{2}{3}(2\pi)^{-1/2}(\zeta(\frac{3}{2}))^4(\zeta(3))^{-1}$.

3. Mean fourth power

In 1922 Hardy and Littlewood [13] proved that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt = 0(T(\log T)^4).$$

This was improved in 1926 by Ingham [18] to

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt = \frac{T}{2\pi^2}(\log T)^4 + 0\ (T(\log T)^3)$$

by using the approximate functional equation for $(\zeta(s))^2$. A simpler proof of this fact was given recently by Ramachandra [24] using the theorem of Montgomery and Vaughan. Subsequently Heath-Brown [14] showed by deep arguments that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt = TP(\log T) + 0(T^{\frac{7}{8}+\varepsilon}),$$

where $P(x)$ is a polynomial of degree 4 in x. The coefficient of x^4 is $1/2\pi^2$ and that of x^3 is

$$2\pi^{-2}[4\gamma - 1 - \log(2\pi) - 12\pi^{-2}\zeta'(2)].$$

He also proved another deep result [16] which asserts that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{12}\,dt = 0(T^2(\log T)^{17}).$$

Previously Atkinson [2] had proved that

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 \exp(-\delta t)\,dt = \frac{1}{\delta} Q\left(\log\frac{1}{\delta}\right) + 0\left(\left(\frac{1}{\delta}\right)^{\frac{8}{9}+\varepsilon}\right)$$

where $Q(x)$ is a polynomial of degree 4 in x. The coefficient of x^4 is $1/2\pi^2$ and that of x^3 is

$$-\frac{1}{\pi^2}\left[2\log(2\pi) - 6\gamma + \frac{24\zeta'(2)}{\pi^2}\right].$$

For the fourth power mean Iwaniec [19] proved the following deep result. Let $I_r (r = 1, 2, \ldots, R)$ be R disjoint intervals of length T_0 satisfying $T^{1/2} \leqslant T_0 \leqslant T$. Then

$$\sum_{r=1}^{R} \int_{I_r} |\zeta(\tfrac{1}{2}+it)|^4 \, dt \ll_\varepsilon (RT_0 + T(R/T_0)^{1/2})T^\varepsilon.$$

Choosing $R = 1, T_0 = T^{2/3}$ we obtain

$$\int_T^{T+T^{2/3}} |\zeta(\tfrac{1}{2}+it)|^4 \, dt \ll T^{\frac{4}{3}+\varepsilon}.$$

The deep result of Iwaniec also implies

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{12} \, dt \ll T^{2+\varepsilon}.$$

This can be seen as follows. It suffices to prove that

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{12} \, dt \ll T^{2+\varepsilon}.$$

The contributions to the twelfth power moment from those t with $|\zeta(\frac{1}{2}+it)| \leqslant T^{\frac{1}{8}+\varepsilon}$ are together $\ll T^{2+9\varepsilon}$ using $\int_0^T |\zeta(\frac{1}{2}+it)|^4 \, dt \ll T(\log T)^4$. So one has to consider the range

$$T^{\frac{1}{8}+\varepsilon} \leqslant |\zeta(\tfrac{1}{2}+it)| \leqslant T^{\frac{1}{6}+\varepsilon}$$

since $\mu(\frac{1}{2}) \leqslant \frac{1}{6}$. We divide the range $[T^{\frac{1}{8}+\varepsilon}, T^{\frac{1}{6}+\varepsilon}]$ into $[T^{\frac{1}{8}+\varepsilon}, 2T^{\frac{1}{8}+\varepsilon})$, $[2T^{\frac{1}{8}+\varepsilon}, 4T^{\frac{1}{8}+\varepsilon})$, $[4T^{\frac{1}{8}+\varepsilon}, 8T^{\frac{1}{8}+\varepsilon}), \ldots$ the last interval either terminates at $T^{\frac{1}{6}+\varepsilon}$ or goes just a little beyond. In any typical interval $[V, 2V)$ we have $V \leqslant |\zeta(\frac{1}{2}+it)| < 2V$. For any given V we consider the set $G(V)$ of all t with $V \leqslant |\zeta(\frac{1}{2}+it)| < 2V$. We choose a small $\delta > 0$ to be fixed later. We next divide $[T, 2T]$ into disjoint intervals $[T, T+V^{4-\delta})$, $[T+V^{4-\delta}, T+2V^{4-\delta})$, $[T+2V^{4-\delta}, T+3V^{4-\delta}), \ldots$ the last interval either terminating at $2T$ or going just a little beyond $2T$. Out of these we retain only those which have at least one point of $G(V)$. With every such interval $[A, B)$ we associate the interval I: $(A-(\log T)^2, B+(\log T)^2)$. The intervals I are not disjoint. But every point of $[A, B)$ is interior to at most two intervals I. Hence if we put $T_0 = V^{4-\delta} + 2(\log T)^2$ we have

$$\sum_I \int_I |\zeta(\tfrac{1}{2}+it)|^4 \, dt \ll (RV^{4-\delta} + T(R/V^{4-\delta})^{1/2})T^\varepsilon.$$

By convexity each of the integrals on the left $\geqslant V^4 (\log T)^{-2}$. Hence

$$RV^4 (\log T)^{-2} \ll (RV^{4-\delta} + T(R/V^{4-\delta})^{1/2}T^\varepsilon.$$

We now put $\delta = 16\varepsilon$. It follows that

$$RV^4 (\log T)^{-2} \ll (R/V^{4-\delta})^{1/2} T^{1+\varepsilon}.$$

Hence $R \ll \dfrac{T^{2+\varepsilon}}{V^{12-\delta}} \ll \dfrac{T^{2+100\varepsilon}}{V^{12}}$. On the other hand by Iwaniec's result we have also

$$\sum_I \int_I |\zeta(\tfrac{1}{2}+it)|^4 \, dt \ll_\varepsilon (RT_0 + T(R/T_0)^{1/2})T^\varepsilon.$$

Here the first term on the right side can be ignored and the left side is

$$\geqslant \sum_I \mu(I) V^4$$

where $\mu(I)$ is the measure of $G(V) \cap I$. Hence

$$\sum_I \mu(I) \ll \frac{1}{V^4} T^{1+\varepsilon} (R/T_0)^{1/2}$$

and so

$$\sum_I \mu(I) V^{12} \ll V^8 T^{1+\varepsilon} (T^{2+100\varepsilon}/V^{16-\delta})^{1/2} \ll T^{2+100\varepsilon}$$

i.e.

$$\int_{G(V)} |\zeta(\tfrac{1}{2}+it)|^{12}\, dt \ll T^{2+100\varepsilon}$$

Summing up over all V the twelfth power moment stated in the beginning is proved.

4. Mean $(2k)$th power

Titchmarsh [36] was the first to prove the following result. For every fixed positive integer k,

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k} \exp(-\delta t)\, dt \gg_k \frac{1}{\delta}\left(\log\frac{1}{\delta}\right)^{k^2}.$$

This gives

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt = \Omega((\log T)^{k^2}).$$

Put

$$M(k,T,T+H) = \frac{1}{H}\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt$$

where k is a positive real number and

$$T \geqslant H \geqslant 100 \log\log T \geqslant C_0$$

where C_0 is a large positive constant. Ramachandra [25] was the first to prove that if $2k$ is a positive integer then $M(k,T,T+H) \gg_k (\log H)^{k^2}$. This gives

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt \gg_k (\log T)^{k^2}.$$

Using Gabriel's convexity theorem this was upheld by Heath-Brown [17] for all rational $k > 0$. Regarding upper bounds Ramachandra [26] was the first to prove that

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|\, dt \ll (\log T)^{1/4}.$$

In fact he [26] proved that $M(\frac{1}{2},T,T+T^{1/2+\varepsilon}) \ll (\log T)^{1/4}$ without any hypothesis and $M(\frac{1}{2},T,T+T^{1/4+\varepsilon}) \ll (\log T)^{1/4}$ on Riemann hypothesis. Heath-Brown [17] proved

again by using Gabriel's convexity theorem that $M\left(\frac{1}{m}, T, 2T\right) \ll (\log T)^{1/m^2}$ where $m \geqslant 1$ is an integer. Jutila [21] proved that

$$(\log T)^{1/m^2} \ll M\left(\frac{1}{m}, T, 2T\right) \ll (\log T)^{1/m^2}$$

where the constants implied by the Vinogradov symbol $\ll$ are independent of m. He applied this to show that the measure of the set $|\zeta(\frac{1}{2}+it)| \geqslant V$ is

$$\leqslant \exp\left(-\frac{(\log V)^2}{\log\log T}\left(1+0\left(\frac{\log V}{\log\log T}\right)\right)\right)$$

where $1 \leqslant V \leqslant \log T$.

Regarding lower bounds Ramachandra [27] was the first to prove

$$M(k, T, T+H) \gg (\log H)^{k^2} (\log\log H)^{-C}$$

where $k > 0$ is any real constant and C depends only on k. Using Heath-Brown's idea he [28] proved

$$M(k, T, T+H) \underset{k}{\gg} (\log H)^{k^2}$$

where $k > 0$ is any rational constant. Using the same idea he proved [29, 30] that for irrational k

$$M(k, T, T+H) \gg g\left(\frac{\log H}{\log\log H}\right)^{k^2},$$

where g is a function of H which tends to infinity with H.

Ramachandra [31] has proved (see also Balasubramanian and Ramachandra [4]) that if k is an integer subject to $1 \leqslant k \leqslant \log H$, we have uniformly in k, T, H

$$M(k, T, T+H) \gg \sum_{n \leqslant H/100} \frac{(d_k(n))^2}{n}\left(1 - \frac{\log n}{\log H} + \frac{1}{\log\log H}\right).$$

Put $\Lambda = (\text{RHS})^{1/2k}$. The quantity Λ has been studied as a function of k by Balasubramanian. It was shown by Balasubramanian and Ramachandra [5] that the maximum of Λ as k varies is attained in $1 \leqslant k \leqslant \log H$ and that $\log \max \Lambda \gg \left(\frac{\log H}{\log\log H}\right)^{1/2}$. By an ingeneous argument Balasubramanian [6] showed that

$$\log \max \Lambda \sim \alpha\left(\frac{\log H}{\log\log H}\right)^{1/2},$$

where $\alpha = \frac{1}{2} \max_{l > 0,\ l \text{ is real}} \{(\exp(-2l) + \int_{2l}^{\infty} \frac{e^{-\theta}}{\theta} d\theta)(2l \exp(2l)^{1/2})\}$.

Richert showed by his pocket calculator that $\alpha = 0.75\ldots$. Thus

$$\max_{T \leqslant t \leqslant T+H} |\zeta(\tfrac{1}{2}+it)| > \exp\left(\frac{3}{4}\left(\frac{\log H}{\log\log H}\right)^{1/2}\right)$$

where $T \geqslant H \geqslant 100 \log\log T$ and $T > C_0$. For the earlier history of the problem see [32] and [5]. This is an improvement on the result

$$\max_{0 \leqslant t \leqslant T} |\zeta(\tfrac{1}{2}+it)| > \exp\left(\frac{1}{20}\left(\frac{\log T}{\log\log T}\right)^{1/2}\right)$$

for $T \geqslant T_0$ due to Montgomery [23] who proved this on the assumption of the Riemann hypothesis.

5. Other important results

Let

$$M(s) = \sum_{m \leqslant M} a_m m^{-s}$$

and

$$S(T, M) = \int_0^T |\zeta(\tfrac{1}{2}+it) M(it)|^2 \, dt.$$

Iwaniec [21] was the first to prove that

$$S(T, M) \ll T^{1+\varepsilon} \sum_{m \leqslant M} |a_m|^2$$

for $M \leqslant T^{1/2}$ unconditionally and for $M \leqslant T^{4/7}$ conditionally. In the case $a_m = \mu(m)$ Heath-Brown has proved this result unconditionally for $M \leqslant T^{8/15}$ (see p. 306 of [8]). In a series of papers Deshouillers and Iwaniec [8, 9] have proved this unconditionally when $a_m = \Lambda(m)$ or $a_m = \mu(m)$ for $M \leqslant T^{4/7}$.

Iwaniec [20] was the first to prove that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 |M(it)|^2 \, dt \ll T^{1+\varepsilon} \sum_{m \leqslant M} |a_m|^2$$

for $M \leqslant T^{1/10}$. Deshouillers and Iwaniec prove this for $M \leqslant T^{1/5}$. Asymptotic formulae are also important (*e.g.* for the discussion of the zeros on the critical line). In this connection Balasubramanian and Conrey [7] have proved that

$$\int_0^T |\zeta(\tfrac{1}{2}+it) M(\tfrac{1}{2}+it)|^2 \, dt \sim T \sum_{m_1, m_2 \leqslant M} \frac{a_{m_1} a_{m_2}}{[m_1, m_2]} \left(\log \frac{T(m_1, m_2)}{2\pi m_1 m_2} + 2\gamma - 1\right)$$

Provided $a_m = \mu(m) f(m)$ where f is a smooth function $\ll 1$ and $M \leqslant T^{\frac{8}{15}-\varepsilon}$. Using this and refining the method of Levinson they proved that 38 % of the zeros lie on the critical line.

In the theory of the Riemann zeta-function an indispensible book is [37]. This has been very useful in the preparation of this article.

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On the ratio of values of a polynomial

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Abstract. For given positive integers a and b, the equation $a(x+1)\dots(x+k) = b(y+1)\dots(y+k)$ in positive integers is considered. More general equations are also considered.

Keyword. Exponential diophantine equations.

1. Introduction

We shall always assume that a, b, x, y and k are positive integers satisfying $b > a$ and $x - y \geqslant k$. We consider the equation

$$a(x+1)\dots(x+k) = b(y+1)\dots(y+k). \tag{1}$$

We re-write (1) as

$$u_k + A_1 u_{k-1} + \dots + A_k u_0 = 0 \tag{2}$$

where

$$u_n = ax^n - by^n, \quad 0 \leqslant n \leqslant k$$

and $A_1, \dots, A_k$ are positive integers given by

$$(X+1)\dots(X+k) = X^k + A_1 X^{k-1} + \dots + A_k.$$

Let α be a real number given by

$$ax^\alpha = by^\alpha.$$

Thus

$$\alpha = \left(\log\frac{b}{a}\right)\left(\log\frac{x}{y}\right)^{-1} > 0.$$

We prove:

THEOREM 1. *There exist effectively computable positive numbers C_1, C_2 and C_3 depending only on a and b such that* (1) *with $k \geqslant C_1$ implies that*

$$C_2 k^2 x^{-1} < k - \alpha < C_3 \eta \tag{3}$$

where

$$\eta = \min(k^2 x^{-1}, x^{-1/3}).$$

In particular, (1) implies

$$0 < k - \alpha < 1, \quad k \geqslant C_1.$$

Thus we have:

COROLLARY 1. *Equation* (1) *implies that either* $k < C_1$ *or* $k = [\alpha + 1]$.

For given positive integers a, b and $k \geqslant 3$ such that the binary form $aX^k - bY^k$ is irreducible over rationals, it follows from a theorem of Schinzel [3] that (1) has only finitely many solutions in x and y. We remark that this result is not effective.

We apply an estimate of Baker [1] on linear forms in logarithms to prove:

THEOREM 2. *Equation* (1) *implies that* $\max(x, y, k)$ *is bounded by an effectively computable number depending only on* a, b *and the greatest prime factor of* xy.

It follows from theorem 1 that (1) implies

$$x \geqslant C_4 k^3$$

where $C_4 > 0$ is an effectively computable number depending only on a and b. Further it follows from Crammer's conjecture on distance between consecutive primes that (1) implies

$$(\log x)^2 \geqslant C_5 k$$

where $C_5 > 0$ is an absolute constant. We apply an inequality of van der Poorten [2] on p-adic linear forms in logarithms to obtain the following result.

THEOREM 3. *Equation* (1) *implies that*

$$\log x \leqslant C_6 k$$

where $C_6 > 0$ *is an effectively computable number depending only on* a, b *and the greatest prime factor of* $y(x-y)$.

We state the following direct consequence of theorem 3.

COROLLARY 2. *Equation* (1) *implies that* $\max(x, y)$ *is bounded by an effectively computable number depending only on* a, b, k *and the greatest prime factor of* $y(x-y)$.

Now we apply an argument of theorem 1 to a more general equation. For positive integers m and H, denote by $S(m, H)$ the set of all polynomials

$$P(X) = B_0 X^m + B_1 X^{m-1} + \ldots + B_m$$

where $B_0 > 0, B_1 > 0$ and $B_2, \ldots, B_m$ are non-negative integers not exceeding H. For positive integers a_1, b_1, x_1 and y_1 with $b_1 > a_1$ and $x_1 > y_1$, let β be a positive real number given by

$$a_1 x_1^\beta = b_1 y_1^\beta.$$

Then we have:

THEOREM 4. *Let* $\delta > 0$ *and* $P \in S(m, H)$. *Let* a_1, b_1, x_1 *and* y_1 *with* $b_1 > a_1$ *and* $x_1 > y_1$ *be positive integers. For* $l_1 \in \mathbf{Z}$ *with*

$$|l_1| \leqslant m^{-1} x_1^{m-1-\delta}.$$

suppose that

$$a_1 P(x_1) = b_1 P(y_1) + l_1. \tag{4}$$

There exist effectively computable positive numbers C_7, C_8 *and* C_9 *depending only on* $\delta, a_1,$

b_1 and H such that for every $m \geqslant C_7$, we have

$$C_8 x_1^{-1} < m - \beta < \min(C_9 x_1^{-1}, 1). \tag{5}$$

Thus (4) implies

$$0 < m - \beta < 1, \quad m \geqslant C_7.$$

By taking $P(X) = X^m + X^{m-1} + \ldots + 1$, we see from theorem 4 that

$$a_1 \frac{x_1^{m+1} - 1}{x_1 - 1} = b_1 \frac{y_1^{m+1} - 1}{y_1 - 1} + l_1$$

with $m \geqslant C_7$ implies (5).

2. Linear forms in logarithms

We shall need the following results on linear forms in logarithms for the proof of theorems 2 and 3. Let $\alpha_1, \ldots, \alpha_n$ be non-zero rational numbers of heights not exceeding $A_1, \ldots, A_n$ respectively, where we assume that $A_j \geqslant 3$ for $1 \leqslant j \leqslant n$. The height of a rational number E/F with $(E, F) = 1$ is defined as $\max(|E|, |F|)$. Put

$$\Omega' = \prod_{j=1}^{n-1} \log A_j \quad \text{and} \quad \Omega = \Omega' \log A_n.$$

THEOREM A (Baker [1]). *There exist effectively computable absolute constants $C_{10} > 0$ and $C_{11} > 0$ such that the inequalities*

$$0 < |\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1| < \exp(-(C_{10} n)^{C_{11} n} \Omega \log \Omega' \log B)$$

have no solution in rational integers $b_1, \ldots, b_n$ of absolute values not exceeding $B(\geqslant 3)$.

THEOREM B (van der Poorten [2]). *Let $p > 0$ be a prime number. There exist effectively computable absolute constants $C_{12} > 0$ and $C_{13} > 0$ such that the inequalities*

$$\infty > \operatorname{ord}_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) > (C_{12} n)^{C_{13} n} \Omega (\log B)^2 \frac{p}{\log p}$$

have no solution in rational integers $b_1, \ldots, b_n$ with absolute values at most $B(\geqslant 3)$.

3. Proof of theorem 1

Denote by $c_1, c_2, \ldots, c_{18}$ effectively computable positive numbers depending only on a and b. We may assume that $k \geqslant c_1$ with c_1 sufficiently large. Then, since $x \neq y$, observe that $u_k u_{k-1} \neq 0$ (see lemma 1 of [4]). Further $c_1 \leqslant k \leqslant x - y < x$ and (1) implies that none of $(x+1), \ldots, (x+k)$ is a prime number. Therefore it follows from the well-known results on difference between consecutive primes* that

$$x \geqslant k^{3/2}. \tag{6}$$

* It is not necessary to use the results on difference between consecutive primes.

Now we sharpen inequality (6). Denote by d the greatest common divisor of $(x+1)\ldots(x+k)$ and $(y+1)\ldots(y+k)$. By (1), we see that $(x+1)\ldots(x+k)/d$ divides b. In particular

$$x^k < (x+1)\ldots(x+k) \leqslant bd.$$

By an argument of Erdos, the contribution in d from primes not exceeding $2k$ is at most

$$k^k(x+k)^{\pi(2k)}$$

Further, by (1), the contribution in d of primes greater than $2k$ does not exceed

$$(x-y+k)^{2k}((2k)!)^{-1}.$$

By (1), we see that $ax^k < b(y+k)^k$ which implies

$$x-y \leqslant c_2 k^{-1}x + 2k. \tag{7}$$

Hence

$$x^k < bd \leqslant (c_3 k)^k x^{\pi(k)}\left(\max\left(\frac{x}{k^2}, 1\right)\right)^{2k}$$

which, together with (6), implies

$$x \geqslant c_4 k^3. \tag{8}$$

Combining (7) and (8), we have

$$x-y \leqslant 2c_2 k^{-1}x. \tag{9}$$

All the summands on the left side of (2) cannot be of the same sign. Therefore, since $x > y$, we see that $u_k > 0$ which, together with (9), implies

$$x-y \geqslant c_5 k^{-1}x. \tag{10}$$

Suppose that $u_{k-1} > 0$. Observe that

$$u_k - x u_{k-1} = b y^{k-1}(x-y).$$

Therefore, by (9) and (10), we see that

$$u_k \geqslant c_6 k^{-1}x^k.$$

Notice that

$$A_n \leqslant \binom{k}{n} k^n \leqslant k^{2n}, \quad 1 \leqslant n \leqslant k. \tag{11}$$

Re-write (1) as

$$u_k + A_1 u_{k-1} = -A_2 u_{k-2} - \ldots - A_k u_0. \tag{12}$$

The absolute value of the left side of (12) is at least

$$\max(u_k, A_1 u_{k-1}) \geqslant c_6 k^{-1}x^k.$$

On the other hand, it follows from (11) and (8) that the absolute value of the right side of (12) is at most

$$c_7 k^4 x^{k-2}.$$

Comparing these estimates, we obtain

$$x \leqslant c_8 k^{5/2}.$$

Now we apply (8) to conclude that $k \leqslant c_9$ which is not possible if $c_1 > c_9$.

Hence $u_k > 0$ and $u_{k-1} < 0$. By continuity, we see that

$$0 < k - \alpha < 1. \tag{13}$$

Re-write (1) as

$$u_k = -A_1 u_{k-1} - A_2 u_{k-2} - \ldots - A_k u_0.$$

Further

$$|u_{k-1}| = -u_{k-1} = b\, y^{k-1}(1-(x/y)^{-(\alpha-k+1)}).$$

Now it follows from (9), (10), (13) and $A_1 = \dfrac{k(k+1)}{2}$ that

$$c_{10}^{-1}(\alpha-k+1)k\,x^{k-1} \leqslant |A_1 u_{k-1}| \leqslant c_{10} k x^{k-1}.$$

Similarly

$$|A_2 u_{k-2}| \leqslant c_{11} k^3 x^{k-2}$$

which, together with (8), implies that

$$|A_2 u_{k-2}| \leqslant c_{12} x^{k-1}.$$

Further, by (11) and (8),

$$|A_3 u_{k-3} + \ldots + A_k u_0| \leqslant c_{13} k^6 x^{k-3} \leqslant c_{14} x^{k-1}.$$

Therefore $u_k \leqslant 2c_{10} k x^{k-1}$. Also, by (9), (10) and (13),

$$c_{15}\left(\frac{k-\alpha}{k}\right)x^k \leqslant u_k \leqslant c_{16}\left(\frac{k-\alpha}{k}\right)x^k.$$

Consequently $k-\alpha < c_{17} k^2 x^{-1} < 1/4$. Now notice that $x^{-k}u_k \geqslant (2c_{10})^{-1} k x^{-1}$ and hence $k-\alpha > c_{18} k^2 x^{-1}$. Thus we have

$$c_{18} k^2 x^{-1} < k-\alpha < c_{17} k^2 x^{-1}$$

which, together with (8), implies (3). This completes the proof of theorem 1.

4. Proof of theorems 2 and 3

Denote by $v_1, \ldots, v_7$ effectively computable positive numbers depending only on a, b and the greatest prime factor of xy. If $k = 1$, then we re-write (1) as

$$ax - by = b - a \neq 0$$

which, by theorem A, implies that $\max(x, y) \leqslant v_1$. Thus we may assume that $k > 1$. Further we apply again theorem A to obtain

$$x - y \geqslant x(\log x)^{-v_2}$$

which, together with (9), implies

$$k \leqslant (\log x)^{v_3}. \tag{14}$$

Let δ be the least non-negative integer such that $u_{k-\delta} \neq 0$. Observe that $\delta \in \{0, 1\}$, since $x \neq y$. By theorem A,

$$|u_{k-\delta}| \geqslant x^{k-\delta}(k \log x)^{-v_4}$$

On the other hand, it follows from (2), (11) and (14) that

$$|u_{k-\delta}| \leqslant x^{k-\delta-1}(\log x)^{v_5}.$$

Comparing these estimates, we obtain

$$x \leqslant k^{v_6}$$

which, together with (14), implies that $\max(x, y, k) \leqslant v_7$. This completes the proof of theorem 2.

Proof of theorem 3

Denote by $v_8, v_9, \ldots$ effectively computable positive numbers depending only on a, b and the greatest prime factor of $y(x-y)$. We may assume that $x \geqslant v_8$ with v_8 sufficiently large, otherwise the assertion follows immediately. Suppose that

$$\log x > k.$$

It follows from (1) that $x-y$ divides

$$\Delta = (b-a)(y+1)\ldots(y+k) \neq 0.$$

Thus, for a prime p dividing $x-y$, we have

$$\mathrm{ord}_p(x-y) \leqslant \mathrm{ord}_p(\Delta).$$

Let $1 \leqslant n_0 \leqslant k$ satisfy

$$\mathrm{ord}_p(y+n_0) \geqslant \mathrm{ord}_p(y+n)$$

for $n = 1, \ldots, k$. By theorem B, we have

$$\mathrm{ord}_p(y+n_0) \leqslant v_9(\log\log x)^3.$$

Therefore

$$\mathrm{ord}_p(x-y) \leqslant \mathrm{ord}_p(\Delta) \leqslant v_9(\log\log x)^3 + v_{10}k.$$

Hence

$$\log(x-y) \leqslant v_{11}((\log\log x)^3 + k).$$

On the other hand we see, by (10) which is also valid for $k < c_1$, that

$$\log(x-y) \geqslant \log x - \log k - v_{12}.$$

Comparing these estimates, we obtain $\log x \leqslant v_{13}k$. This completes the proof of theorem 3.

5. Proof of theorem 4

In the proof of theorem 4, we omit some details which already appear in the proof of theorem 1. Denote by $c_{19}, c_{20}, \ldots$ effectively computable positive numbers depending

only on δ, a_1, b_1 and H. We may assume that $m \geqslant c_{19}$ with c_{19} sufficiently large. Let $P \in S(m, H)$ be given by

$$P(X) = B_0 X^m + B_1 X^{m-1} + \ldots + B_m.$$

Re-write (4) as

$$B_0 U_m + B_1 U_{m-1} + \ldots + B_m U_0 = l_1 \tag{15}$$

where

$$U_n = a_1 x_1^n - b_1 y_1^n; \quad 0 \leqslant n \leqslant m.$$

Observe that (4) implies

$$c_{20} m^{-1} x_1 < x_1 - y_1 < c_{21} m^{-1} x_1. \tag{16}$$

Observe that

$$U_m - x_1 U_{m-1} = b_1 y_1^{m-1} (x_1 - y_1)$$

which implies

$$\max(|U_m|, |U_{m-1}|) \geqslant c_{22} m^{-1} x_1^{m-1}.$$

Thus all the summands on the left side of (15) cannot be of the same sign. Then, since $x_1 > y_1$, we see that $U_m > 0$. Suppose $U_{m-1} > 0$. Then

$$U_m \geqslant c_{23} x_1^{m-1}.$$

Further (15) implies that $x_1 \leqslant c_{24}$ which, by (4), is not possible if c_{19} is sufficiently large.

Hence $U_m > 0$ and $U_{m-1} < 0$. Thus

$$0 < m - \beta < 1. \tag{17}$$

Let $N = [(\log m)^2]$. Using (16) and (17), we have

$$c_{25}\left(\frac{m-\beta}{m}\right) x_1^m < U_m < c_{26}\left(\frac{m-\beta}{m}\right) x_1^m,$$

$$c_{27}\left(\frac{\beta - m + 1}{m}\right) x_1^{m-1} < |U_{m-1}| < c_{28} m^{-1} x_1^{m-1},$$

$$|U_{m-r}| \leqslant c_{29} r m^{-1} x_1^{m-r}, \quad 2 \leqslant r \leqslant N$$

and

$$|U_{m-r}| \leqslant c_{30} x_1^{m-r}, \quad N < r \leqslant m.$$

In view of these estimates, (15) implies (5). This completes the proof of theorem 4.

Remark

In theorem 4, the restriction on $|l_1|$ can be relaxed to $|l_1|$ does not exceed constant times $m^{-1} x_1^{m-1}$.

Acknowledgement

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Linear and non-linear magnetoconvection in a porous medium

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Abstract. Linear and non-linear magnetoconvection in a sparsely packed porous medium with an imposed vertical magnetic field is studied. In the case of linear theory the conditions for direct and oscillatory modes are obtained using the normal modes. Conditions for simple- and Hopf-bifurcations are also given. Using the theory of self-adjoint operator the variation of critical eigenvalue with physical parameters and boundary conditions is studied. In the case of non-linear theory the subcritical instabilities for disturbances of finite amplitude is discussed in detail using a truncated representation of the Fourier expansion. The formal eigenfunction expansion procedure in the Fourier expansion based on the eigenfunctions of the corresponding linear stability problem is justified by proving a completeness theorem for a general class of non-self-adjoint eigenvalue problems. It is found that heat transport increases with an increase in Rayleigh number, ratio of thermal diffusivity to magnetic diffusivity and porous parameter but decreases with an increase in Chandrasekhar number.

1. Introduction

The linear and non-linear magnetoconvection in an electrically conducting fluid has been extensively studied by many authors ([1], [2], [17], [18], [21], [22]) but the corresponding problem in a porous medium has not been given much attention, inspite of its various engineering and geophysical applications. Rudraiah [12], Rudraiah and Prabhamani [13], and Prabhamani and Rudraiah [9–11], have considered this problem using a universal stability analysis (*i.e.*, stability for arbitrary disturbances) which gives, as in infinitesimal disturbances, only the criterion for the onset of convection but is silent about the prediction of heat transfer. The study of heat transfer is essential to understand the mechanism of heat transfer from the deep interior of the earth to shallow depth in geothermal region. In other words, the problem considered in this paper is similar to a situation that exists in the geothermal regions where the surface liquid possesses a general upward convective drift due to the buoyancy induced by Joule heat and interior temperature. Since the rising liquid is cooled as it approaches the surfaces where heat is removed by evaporation, radiation and movement in surface streams an unstable state may be induced and complicated convective motions appear in the layers near the surface. Therefore convection through a porous medium in the presence of a geothermal magnetic field is studied in this paper because the results of such a study is useful in understanding the field behaviour in the extraction of energy in the geothermal regions.

2. Mathematical formulation

The physical configuration considered in this paper is shown in figure 1 which consists of a horizontal thin porous layer of permeability k_p and of infinite extent filled with a

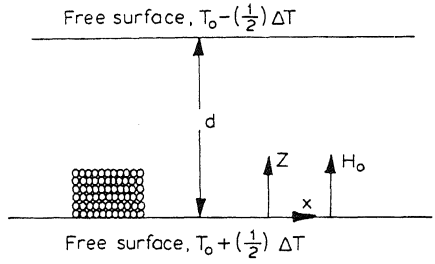

Figure 1. Physical configuration.

conducting Boussinesq fluid heated from below and permeated by an externally applied uniform vertical magnetic field H_0. The layer has thickness d and is bounded by two free surfaces. The upper surface is at a constant temperature $T_0 - \Delta T/2$ and the lower at $T_0 + \Delta T/2$. We write the total temperature as

$$T_{\text{total}} = T_0 - \Delta T(z/d - 1/2) + T(x, y, z, t)$$

where $T(x, y, z, t)$ is the deviation of the temperature from the linear profile and we assume that all the physical quantities are independent of y. In other words, we consider here the two-dimensional horizontal rolls.

For a creeping electrically conducting Boussinesq fluid flowing through a porous medium, the momentum equation is the modified Darcy's law

$$\frac{\partial \mathbf{q}}{\partial t} = -\frac{1}{\rho_0}\nabla P + \frac{\rho}{\rho_0}\mathbf{g} - \frac{\nu}{k_p}\mathbf{q} + \frac{\mu_m}{\rho_0}(\mathbf{H}\cdot\nabla)\mathbf{H}, \tag{1}$$

where $\mathbf{q}$ is the Darcy velocity, ρ is the density, ρ_0 is the density at the reference temperature $T = T_0$, $\mathbf{H}$ is the magnetic field, P is the total pressure, ν is the kinematic viscosity, μ_m is the magnetic permeability and $\mathbf{g}$ is the acceleration due to gravity. This equation neglects non-linear inertia effects. At high Reynolds number this equation should be generalized to include inertia effects. The effect of inertia must appear as a drag proportional to the square of the velocity and is of the form

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho_0}\nabla P + \frac{\rho}{\rho_0}\mathbf{g} - \frac{\nu}{k_p}\mathbf{q} - \frac{c_b}{k_p^{1/2}}|\mathbf{q}|\mathbf{q} + \frac{\mu_m}{\rho_0}(\mathbf{H}\cdot\nabla)\mathbf{H}, \tag{2}$$

where c_b is a non-dimensional constant and has a more or less universal value for a particular family of materials. This neglects shear effects. In a sparsely distributed porous medium made of spherical particles of uniform size, the porosity is very high and we have to include the usual shear effects produced by the distortion of velocity. In that case, considered in this paper, the basic momentum equation is the modified Brinkman equation

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho_0}\nabla P + \frac{\rho}{\rho_0}\mathbf{g} - \frac{\nu}{k_p}\mathbf{q} - \frac{c_b}{k_p^{1/2}}|\mathbf{q}|\mathbf{q} + \frac{\mu_m}{\rho_0}(\mathbf{H}\cdot\nabla)\mathbf{H} + \nu_m\nabla^2\mathbf{q}. \tag{3}$$

In this paper since we deal with pure magnetoconvection problem, we neglect the drag

proportional to the square of the velocity and consider the basic equations:

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho_0}\nabla P + \frac{\rho}{\rho_0}\mathbf{g} - \frac{\nu}{k_p}\mathbf{q} + \nu\nabla^2\mathbf{q} + \frac{\mu_m}{\rho_0}(\mathbf{H}\cdot\nabla)\mathbf{H}, \tag{4}$$

$$\nabla\cdot\mathbf{q} = 0, \tag{5}$$

$$\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{H} = (\mathbf{H}.\nabla)\mathbf{q} + \nu_m\nabla^2\mathbf{H}, \tag{6}$$

$$\rho = \rho_0[1 - \beta(T - T_0)], \tag{7}$$

$$\frac{\partial T}{\partial t} + (\mathbf{q}\cdot\nabla)T + (\Delta T/d)w = K_d\nabla^2 T, \tag{8}$$

where β is the volumetric expansion coefficient, ν_m is the magnetic viscosity and K_d is the thermal diffusivity.

These equations, using

$$u = \frac{\partial\psi}{\partial z},\ w = -\frac{\partial\psi}{\partial x},\ H_x = \frac{\partial\phi}{\partial z},\ H_z = -\frac{\partial\phi}{\partial x}$$

and making the resulting equations dimensionless, take the form

$$\frac{\partial\eta}{\partial t} = J(\psi,\eta) - \frac{\mathrm{Pr}Q}{S}J(\phi,\xi) - \mathrm{Pr}\,\mathrm{Ra}\frac{\partial T}{\partial x} + \frac{\mathrm{Pr}Q}{S}\nabla^2(\partial\phi/\partial z) - \frac{\mathrm{Pr}}{P_l}\eta + \mathrm{Pr}\nabla^2\eta, \tag{9}$$

$$\frac{\partial\phi}{\partial t} = J(\psi,\phi) + \frac{\partial\psi}{\partial z} + \frac{1}{S}\nabla^2\phi, \tag{10}$$

$$\frac{\partial T}{\partial t} = J(\psi,T) - \frac{\partial\psi}{\partial x} + \nabla^2 T, \tag{11}$$

where the equations are made dimensionless using d, d^2/K_d, K_d/d, ΔT and H_0 the scales for length, time, velocity, temperature and magnetic field respectively, $\mathrm{Ra} = (\beta g\Delta T d^3)/K_d$ is the Rayleigh number, $\mathrm{Pr} = \nu/K_d$ is the Prandtl number, $P_l = k_p/d^2$ is the porous parameter, $Q = \mu_m H_0^2 d^2/\rho_0\nu\nu_m$ is the Chandrasekhar number, $S = K_d/\nu$, $\eta = \nabla^2\psi$, $\xi = \nabla^2\phi$ and $J(.,.)$ is the Jacobian.

The boundary conditions for the problem, when the boundaries $z = 0$ and $z = 1$ taken as flat, are the magnetic and velocity stress free,

$$\psi = 0, \frac{\partial^2\psi}{\partial z^2} = 0, T = 0, \frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \tag{12}$$

3. Linear stability analysis

In this section, following the analysis of Rudraiah and coworkers [15, 18], we discuss the linear stability analysis considering both the marginal and overstable states. The solutions of this analysis are of great importance in the finite amplitude study, using the local non-linear stability analysis.

The required stability equation for this case, after neglecting the Jacobian terms in (9) and (10) and eliminating ϕ and T, is

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)\left(\frac{\partial}{\partial t}+\frac{1}{S}\nabla^2\right)\left(\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}+\left(\frac{1}{P_l}-\nabla^2\right)\right)\nabla^2\psi$$
$$=\left(\frac{\partial}{\partial t}-\nabla^2\right)\frac{Q}{S}\frac{\partial^2\nabla^2\psi}{\partial z^2}+\left(\frac{\partial}{\partial t}-\frac{1}{S}\nabla^2\right)\mathrm{Ra}\frac{\partial^2\psi}{\partial x^2}. \tag{13}$$

We look for the solution of the form

$$\psi=\exp(p_0 t)\sin\pi\alpha x\sin\pi z, \tag{14}$$

where p_0 $(=p_{0r}+ip_{0i})$ is the frequency. Substituting (14) into (13) and after some simplification, we obtain the dispersion relation

$$p_0^3+\left[\mathrm{Pr}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right)+\frac{\pi^2(\alpha^2+1)}{S}+\pi^2(\alpha^2+1)\right]p_0^2$$
$$+\left[\mathrm{Pr}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right)\pi^2(\alpha^2+1)+\frac{\pi^4(\alpha^2+1)^2}{S}\right.$$
$$\left.+\frac{\mathrm{Pr}Q\pi^2}{S}-\frac{\mathrm{Ra}\alpha^2\mathrm{Pr}}{\alpha^2+1}+\frac{\mathrm{Pr}}{S}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right)\pi^2(\alpha^2+1)\right]p_0$$
$$+\left[\frac{\mathrm{Pr}}{S}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right)\pi^4(\alpha^2+1)^2-\frac{\mathrm{Pr}}{S}\pi^2(\alpha^2+1)\right.$$
$$\left.\times\left(\frac{\mathrm{Ra}\alpha^2}{\alpha^2+1}-Q\pi^2\right)\right]=0. \tag{15}$$

3.1 *Marginal state*

Linear theory predicts the condition for the onset of convection at the critical Rayleigh number (see [2]). The unstable thermal stratification tends to produce steady convection at $\mathrm{Ra}=\mathrm{Ra}_c^m$, the critical Rayleigh number, given by

$$\mathrm{Ra}_c^m=R_L^m+R_M^m, \tag{16}$$

where
$$R_L^m=\frac{\pi^2(\alpha^2+1)^2}{\alpha^2 P_l} \tag{17}$$

is the Lapwood [6] Rayleigh number and

$$R_M^m=\frac{\pi^2(\alpha^2+1)}{\alpha^2}[\pi^2(\alpha^2+1)^2+Q] \tag{18}$$

is the Rayleigh number for magnetoconvection given by Chandrasekhar [2] and the superscript m denotes the marginal state. The critical wave number for the direct mode is obtained from the cubic equation

$$x^3+\frac{1}{2\pi^2}\left(3\pi^2+\frac{1}{P_l}\right)x^2-\left\{\frac{\pi^2+Q}{2\pi^2}+\frac{1}{P_l\pi^2}\right\}=0, \tag{19}$$

with $x=\alpha^2$. Equation (16) reveals that the effects of permeability and magnetic field are

to suppress steady convection. Similarly (19) shows that the wave number x is influenced by the permeability parameter P_l and the Chandrasekhar number Q. Therefore, both permeability and magnetic field influence the cell pattern.

3.2 *Overstability*

The striking similarity that exists on several occasions between the results pertaining to a fluid in the presence of a magnetic fluid and a porous medium makes one to anticipate overstable motions in the present study. Therefore, in this section, we discuss infinitesimal overstable motions.

The condition for overstable motion is that the product of the coefficient of p_0^2 and p_0 in (15) must be equal to the coefficient of p_0^0 (*i.e.*, constant term) and is given by

$$\mathrm{Ra}_c^0 = \frac{\pi^2(\alpha^2+1)(1+S)[(\mathrm{Pr}S/P_l\pi^2(\alpha^2+1))+1]}{\alpha^2\mathrm{Pr}S^2} \times\left[\pi^2(\alpha^2+1)^2+\frac{\mathrm{Pr}QP_l\pi^2(\alpha^2+1)}{(P_l\pi^2(\alpha^2+1)+\mathrm{Pr})(1+S)}\right]. \tag{20}$$

Also, for overstable motions the coefficient of p_0 in (15) should be real and positive, and hence it leads to the condition

$$Q \geqslant \frac{\pi^2(\alpha^2+1)^2(1+(\mathrm{Pr}/P_l\pi^2(\alpha^2+1)))}{\mathrm{Pr}(S-1)} \tag{21}$$

for overstability which is valid only when $S > 1$ *i.e.*, Pm > Pr. The minimum values of Ra_c^0 and the corresponding values of α^2 for given values of Q, P_l, S and Pr = 1 are obtained by varying the values of α^2 in (20) and satisfying the condition (21).

We note that this analysis reduces to the Darcy case if $\pi^2(\alpha^2+1) \ll (1/P_l)$ *i.e.*, neglecting $\pi^2(\alpha^2+1)$ compare to $(1/P_l)$ in $((1/P_l)+\pi^2(\alpha^2+1))$ of equation (15). Similarly it reduces to magnetoconvection in the absence of a porous medium if $1/P_l \ll \pi^2(\alpha^2+1)$ *i.e.*, neglecting $1/P_l$ compare to $\pi^2(\alpha^2+1)$ in $(1/P_l)+\pi^2(\alpha^2+1)$ of (15). This is the same as the limiting case of $P_l \to \infty$ in (15).

3.3 *Bifurcations from the static solution*

The dispersion relation (12) may be used to examine the bifurcations from the static solution. Since this dispersion relation is a cubic polynomial in p_0 with either zero or two sign changes. Hence, by Descarte's rule of signs, either there are no bifurcation points or there are two. In the latter case there will be two types of bifurcation; simple—(or direct) and Hopf bifurcation.

We first seek the condition for a simple-bifurcation for which $p_0 = 0$. Traditionally, this is known as an exchange of stabilities. To obtain the condition we rewrite (12) in the form

$$p_0^3+\left\{\mathrm{Pr}\left(1+\frac{1}{P_l'}\right)+1+\frac{1}{S}\right\}\delta^2p_0^2+\left[\left\{\mathrm{Pr}\left(1+\frac{1}{S}\right)\left(1+\frac{1}{P_l'}\right)+\frac{1}{S}\right\}\delta^4 - \mathrm{Ra}\,\mathrm{Pr}\frac{\pi^2\alpha^2}{\delta^2}+\frac{Q\mathrm{Pr}\pi^2}{S}\right]p_0+\frac{\mathrm{Pr}}{S}\left[\left(1+\frac{1}{P_l'}\right)\delta^6+Q\delta^2\pi^2-\pi^2\alpha^2\mathrm{Ra}\right]=0 \tag{22}$$

where $\delta^2 = \pi^2(\alpha^2+1)$, $P_l' = P_l\delta^2$.

From (22) $p_0 = 0$ when $\mathrm{Ra} = \mathrm{Ra}^{(e)}$, where

$$\mathrm{Ra}^{(e)} = [\delta^6(1+1/P_l')+\pi^2\delta^2 Q]/\pi^2\alpha^2 \tag{23}$$

When $Q = 0$, $\mathrm{Ra}^{(e)} = \mathrm{Ra}_0$, where

$$\mathrm{Ra}_0 = \frac{\delta^6}{\pi^2\alpha^2}(1+1/P_l'). \tag{24}$$

The first term on the right side is the boundary layer correction arising due to the use of modified Brinkman equation. It will prove convenient, especially when we come to consider non-linear solutions to introduce a more compact notation by defining a normalised Rayleigh number

$$r = \mathrm{Ra}\frac{\pi^2\alpha^2}{\delta^6};$$

$$\omega = p_0/\delta^2; \quad \eta' = 1+1/P_l'$$

$$q = Q\pi^2/\delta^4;$$

So that (22) simplifies to

$$\omega^3 + \left\{\mathrm{Pr}\eta' + 1 + \frac{1}{S}\right\}\omega^2 + \left[\mathrm{Pr}\left(\eta' - r + \frac{q}{S}\right) + \frac{1}{S}(\mathrm{Pr}\eta' + 1)\right]\omega$$

$$+\frac{\mathrm{Pr}}{S}(\eta' + q - r) = 0. \tag{25}$$

The condition for simple bifurcation is now $r = r^{(e)}$, where

$$r^{(e)} = \eta' + q. \tag{26}$$

Of the three roots of (25), one is always real and negative. The other pair may be real, or complex conjugates. In the latter case there is a Hopf bifurcation from the static solution (when $\mathrm{Re}(\omega) = 0$, $\omega = \pm i\omega_0$) at $r = r^{(0)}$, where

$$r^{(0)} = \eta' + \frac{(\mathrm{Pr}+1)}{\mathrm{Pr}S(\mathrm{Pr}\eta'+1)}\left(1 + \mathrm{Pr}\eta' + \frac{1}{S}\right) + \frac{(\mathrm{Pr}\eta' + 1/S)q}{S(\mathrm{Pr}\eta'+1)} \tag{27}$$

provided that

$$\omega_0^2 = \frac{\mathrm{Pr}}{S(1+\mathrm{Pr}\eta'+1/S)}(r^{(e)} - r^{(0)}) = \frac{(\mathrm{Pr}+1)}{(\mathrm{Pr}\eta'+1)S^2} + \frac{(S-1)\mathrm{Pr}q}{S^2(\mathrm{Pr}\eta'+1)} \tag{28}$$

is positive.

Hence, if a Hopf bifurcation is possible it always occurs at a lowest value of r than the simple bifurcation. In traditional terms, overstability sets in at $r = r^{(0)} < r^{(e)}$. It is clear from (28) that a necessary condition for the existence of a Hopf bifurcation is

$$S > 1, q > q_0 \equiv \frac{(\mathrm{Pr}+1)}{\mathrm{Pr}(S-1)}. \tag{29}$$

However, for $S < 1$ or $q < q_0$, the only bifurcation from the static solution occurs at $r = r^{(e)}$ and the static solution is unstable for all $r > r^{(e)}$. If, on the other hand, (29) is satisfied, so that $\omega_0^2 > 0$, the static solution is unstable for all $r > r^{(0)}$. The behaviour of

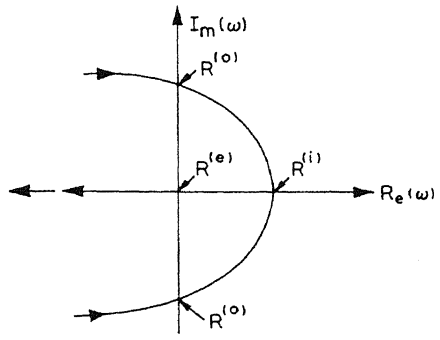

Figure 2. Behaviour of the eigen values when convection sets in *via* a Hopf bifurcation at $R = R^{(0)}$. The arrows indicate the evolution of the eigenvalues ω with increasing R in the complex plane.

the eigenvalues in the complex plane is illustrated in figure 2; as r passes through $r^{(0)}$, the real part of the complex conjugate pair changes sign. As r further increases, $\mathrm{Re}(\omega)$ increases while $|\mathrm{Im}(\omega)|$ decreases. At $r = r^{(i)}$, say $\mathrm{Im}(\omega) = 0$, if the two roots coalesce. Thereafter both eigenvalues remain on the real axis; one increases with increasing r, but the other decreases and passes through zero at $r = r^{(e)}$. In linear theory, therefore, the transition from oscillatory to direct instability occurs at $r = r^{(i)}$. The condition for (25) to have two equal roots is

$$\left(\frac{A_1}{3} - \frac{A_2^2}{9}\right)^3 + \left\{\frac{1}{6}(A_1 A_2 - 3A_3) - \frac{1}{27}A_2^3\right\}^2 = 0$$

where $$A_1 = \mathrm{Pr}\eta' + 1 + \frac{1}{S}$$

$$A_2 = \mathrm{Pr}\left(\eta' - r + \frac{q}{S}\right) + \frac{1}{S}(\mathrm{Pr}\eta' + 1)$$

$$A_3 = \frac{\mathrm{Pr}}{S}(\eta' + q - r).$$

Values of $r^{(i)}$, if needed, must in general be found numerically.

3.4 *Parametric perturbation methods*

The analysis of the previous sections pertain only to the case when the boundaries are isothermal (*i.e.* boundary condition on temperature is of first kind). In many practical problems discussed in §1, the realistic boundary condition on temperature is of the radiation type (*i.e.* boundary condition on temperature is of the third kind). Therefore the effect of radiation boundary condition on the eigenvalues is studied in this section without going into the detailed numerical computation. Using the concept of self adjoint operator we try to find the effect of Biot number, Chandrasekhar number and porous parameter on the monotonicity of Rayleigh number. Our main object here is to show that under what conditions the critical Rayleigh number for marginal convection is an upper bound for finite Biot number and to find an analytical relation for the variation of Rayleigh number with respect to Chandrasekhar number. As an extra

result we also obtain expressions for obtaining first order effects on the eigen-functions. We employ a method which has been used by Narayanan *et al* [7] in a more general context. The first order effects can be calculated using a modified Greens matrix as the kernel of a matrix differential operator.

For this purpose we define an operator L as follows:

$$\mathrm{L} \equiv \begin{bmatrix} (D^2-\pi^2\alpha^2)\{(D^2-\pi^2\alpha^2)-P_l^{-1}\} & \mathrm{Rm}\,(D^2-\pi^2\alpha^2)D & -\mathrm{Ra}^0\pi\alpha \\ -\mathrm{Rm}\,(D^2-\pi^2\alpha^2)D & -\dfrac{\mathrm{Rm}}{S}(D^2-\pi^2\alpha^2)^2 & 0 \\ -\mathrm{Ra}^0\pi\alpha & 0 & \mathrm{Ra}^0(D^2-\pi^2\alpha^2) \end{bmatrix}$$

where $D = \dfrac{\partial}{\partial z}$ and $\mathrm{Rm} = Q/S$. (30)

We define a vector v such that

$$\mathbf{v}^t = [\psi, \phi, T],$$

where the three vectors represent the dependent variables. Then from (4) to (8) after neglecting the Jacobians, we have

$$L\mathbf{v} = 0. \tag{31}$$

To visualize the variation of Ra^0 with respect to Q, we differentiate (31) with respect to Q and obtain

$$L\hat{\mathbf{v}} = \mathfrak{h}, \tag{32}$$

where

$$\mathfrak{h}^t = \left[-\frac{1}{S}(D^2-\pi^2\alpha^2)D\phi + \hat{\mathrm{Ra}}^0\pi\alpha T, 0, 0\right] \tag{33}$$

and '$\wedge$' represents the derivative with respect to Q.

Define a suitable inner product between two vectors **a** and **b** as $\langle \mathbf{a}, \mathbf{b} \rangle$ such that

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_v \mathbf{a}^{*t} \cdot \mathbf{b}\, \mathrm{d}V, \tag{34}$$

where V is the domain of the integral operator in which **a** and **b** are defined and the asterisks represent complex conjugates. We can easily see that L is selfadjoint and the boundary conditions are also selfadjoint. Thus applying a Fredholm alternative condition to (32) we get

$$\pi\alpha Q\left\{\hat{\mathrm{Ra}}^0 - \frac{\mathrm{Ra}^0}{Q}\right\}\int_v \psi^* T\,\mathrm{d}v = -\int_v |D^2\psi|^2\,\mathrm{d}v$$

$$-(2\pi^2\alpha^2 + P_l^{-1})\int_v |D\psi|^2\,\mathrm{d}V - \pi^2\alpha^2(\pi^2\alpha^2 + P_l^{-1})\int_v |\psi|^2\,\mathrm{d}v. \tag{35}$$

We use the energy and continuity equation to arrive at the conclusion that $\hat{\mathrm{Ra}}^0 > 0$ if Ra^0 is positive. From this it is clear that an increase in strength of the magnetic field causes an increase in Ra^0 if we have the most unstable case.

We can consider the dependence of Ra^0 on the porous parameter by taking the

derivatives of (31) with respect to P_l and obtain

$$L\bar{\mathbf{v}} = \bar{\mathbf{h}}, \tag{36}$$

where

$$\bar{\mathbf{h}}^t = [(D^2 - \pi^2\alpha^2)\psi + \overline{\mathrm{Ra}^0}\,\pi\alpha T, 0, 0] \tag{37}$$

and overbars $(-)$ represent differentiation with respect to P_l. On applying the solvability condition we get

$$\pi\alpha\overline{\mathrm{Ra}^0}\int_v \psi^* T\,\mathrm{d}v = \int_v |D\psi|^2\,\mathrm{d}v + \pi^2\alpha^2\int_v |\psi|^2\,\mathrm{d}v. \tag{38}$$

This yields the condition that Ra^0 increases if P_l decreases.

Finally, we consider the effect of boundary parameter, the Biot number Bi, on Ra^0 we shall see that the obtained Rayleigh numbers are upper bounds for finite Biot number. We note that the effect of surface tension gradient is not considered here.

We may differentiate (31) and the corresponding boundary conditions with respect to Bi and obtain

$$L\tilde{\mathbf{V}} = [\tilde{\mathrm{R}}\mathrm{a}^0\,\pi\alpha T, 0, 0]^t \tag{39}$$

where tilde $(\sim)$ overbars represent the differentiation with respect to Bi and at the boundary we have

$$D\tilde{T} + Bi\,\tilde{T} + T = 0. \tag{40}$$

Thus on applying the solvability condition to (39) in the light of (40), we get after much algebraic manipulation

$$\pi\alpha\frac{\tilde{\mathrm{R}}\mathrm{a}^0}{\mathrm{Ra}^0}\int_v \psi^* T\,\mathrm{d}v = \int_v (-|T|^2)\,\mathrm{d}v. \tag{41}$$

On using the energy equation, we get from (41)

$$\frac{\partial \ln \mathrm{Ra}^0}{\partial Bi} > 0.$$

Thus if $\mathrm{Ra}^0 < 0$ (such as the case of heated from above) then Ra^0 will decrease with an increase in Bi. In the present problem we are concerned with the case of heated from below and so the case of Bi tending to infinity provides an upper bound on Ra^0. Equation (39) is a vehicle for calculating the first-order effects and a modified Green matrix is needed for this calculation. Such approximations are of a practical value.

3.5 *A completeness theorem*

The non-linear magnetoconvection in a porous medium studied in the next section is based on expanding the Fourier components of a spatially periodic disturbance in the eigen-functions of the corresponding linear stability problem. This is, usually, known as Stuart's shape assumption. For a general discussion of this procedure and an application to the problem of the stability of plane parallel flow and of viscous flow between rotating concentric cylinders see Diprima and Habetler [4]; for an application to the stratified viscous shear flows problem see Herron [5]. Their completeness

theorem is the generalization of Naimark's theorem [8]. In this section following Diprima and Habetler [4] we will justify the formal eigenfunction expansion procedure used in the next section by proving a completeness theorem for a general class of non-selfadjoint eigenvalue problems which includes the matrix differential operators.

It is not difficult to show that the system (4) to (8), neglecting the Jacobian terms and using the solution of the form

$$f(z)\exp\{i\alpha(x-ct)\},$$

may be written as

$$L\Phi = \lambda M\Phi, \tag{42}$$

where

$$L = \begin{bmatrix} L_0 - \dfrac{Q}{S}D^2(D^2-\alpha^2) & -\dfrac{i\alpha \mathrm{Ra}}{S}(D^2-\alpha^2) \\ i\alpha & -(D^2-\alpha^2) \end{bmatrix},$$

$$M = \begin{bmatrix} L_1 & \alpha \mathrm{Ra} \\ 0 & 1 \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \psi \\ T \end{bmatrix},$$

$$L_0 = \left[-\frac{\alpha^2 C^2}{\mathrm{Pr}} - \frac{1}{S}(D^2-\alpha^2)\left\{\frac{1}{P_l} - (D^2-\alpha^2)\right\}\right](D^2-\alpha^2)$$

$$L_1 = \left[\frac{1}{P_l} - (D^2-\pi^2\alpha^2) - \frac{1}{\mathrm{Pr}S}(D^2-\pi^2\alpha^2)\right](D^2-\pi^2\alpha^2),$$

$$D = \mathrm{d}/\mathrm{d}z \quad \text{and} \quad \lambda = i\alpha C.$$

This is the type of non-standard non-selfadjoint eigenvalue problem studied by Diprima and Habetler [4] and Herron [5]. The theorem that we try to establish concerns eigenvalue problems of the form (42), in a Hilbert space H where L and M are matrix differential operators with $dmn\ L \subset dmn\ M$, and M is positive definite. Another Hilbert space H_M is embedded in H where $[f, g] = \langle f, Mg \rangle$ (for $f, g \in dmn\ M$) is the corresponding inner product and $[f,f]^{1/2} = \| f \|_M$. We have the following theorem

Theorem: Let $L = L_s + L_B$ (43)

where

$$L_s = \begin{bmatrix} \dfrac{1}{S}(D^2-\alpha^2)^3 & 0 \\ 0 & -(D^2-\alpha^2) \end{bmatrix} \tag{44}$$

$$L_B = \begin{bmatrix} -\dfrac{\alpha^2 C^2}{\mathrm{Pr}} - \dfrac{1}{S\mathrm{Pr}}(D^2-\alpha^2)^2 - \dfrac{Q}{S}(D^2-\alpha^2)D^2 & -\dfrac{i\alpha \mathrm{Ra}}{S}(D^2-\alpha^2) \\ i\alpha & 0 \end{bmatrix} \tag{45}$$

are linear operators with $dmn\ L_s = dmn\ L_B \subset dmn\ M$ dense in a Hilbert space H such that

(i) M and L_s are positive bounded below with $rng\ M = rng\ L_s$.

(ii) $M^{-1}L_B$ is bounded in H_M and *dmn* L_s is dense in H_M.
(iii) G^{-1} is compact and *rng* $G^{-1} \subset$ *dmn* M where G^{-1} is the inverse of the selfadjoint extension of $M^{-1}L_s$ in H_M.
(iv) For some γ, *rng* $(L_s + L_B + \gamma M) =$ *rng* M and $(G + M^{-1}L_B + \gamma I)$ has an inverse.
(v) There exists a sequence of concentric circles $\{C_k\}$ with radii $\{r_k\}$ such that

(a) $\lim_{k\to\infty} r_k = \infty$.

(b) $d[c_k, \sigma\theta(L_s, M)] > 0$ for all k.

(c) $\lim_{k\to\infty} d[c_k, \sigma\theta(L_s, M)] = \infty$

Then (i), the eigenvalues of (42) lie within the circles of radii $\|M^{-1}L_B\|$ about those of $L_s = \lambda M\Phi$, and (ii) the generalised eigenfunctions of (42) span H_M such that for $\Phi \in H_M$

$$\phi = \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} a_{ij}\phi_{ij},$$

where $\phi_{k_1}, \phi_{k_2}, \ldots, \phi_{k_{n_k}}$ are eigenfunctions corresponding to eigenvalues in $r_{k-1} < |\gamma| < r_k$ for $k > 1$ and $|\gamma| < r_1$ for $k = 1$.

Following Diprima and Habetler [4] with L_s given by (44) it is straight forward to show that the hypothesis (i), (iii) and (v) are satisfied. If $\|M^{-1}L_B\|$ is bounded hypothesis (ii) holds and (iv) follows. In (45) L_B is different from the one given by Diprima and Habetler [4] and also of Herron [5]. However following [5] it is easy to show that $M^{-1}L_B$ is bounded. Hence all the hypotheses of Diprima and Habetler [4] is satisfied and its conclusions justify the use of spatial expansion in the non-linear problem based on the eigenfunctions of the linear stability problem.

4. Finite amplitude analysis with a limited representation

In this section we discuss, following Rudraiah and Vortmeyer [14] the finite amplitude analysis by considering a truncated representation of Fourier series for velocity, magnetic and temperature fields and try to understand the physics of the problem with minimum mathematics. We note that the results obtained from such a simple analysis can be used as a starting value in solving a general non-linear convection problem.

The first effect of non-linearity is to distort the temperature field through the interaction of ψ and T and the zonal current field through the interaction of ψ and ϕ. The distortion of temperature field will correspond to a change in the horizontal mean, *i.e.*, a component of the form $\sin 2\pi z$ will be generated. Similarly, zonal current field will be distorted by a component of the form $\sin 2\pi\alpha x$. Thus, a minimal system which describes finite amplitude convection is given by

$$\psi = A(t)\sin\pi\alpha x \sin\pi z, \tag{46}$$

$$T = B(t)\cos\pi\alpha x \sin\pi z + C(t)\sin 2\pi z, \tag{47}$$

$$\phi = D(t)\sin\pi\alpha x \cos\pi z + E(t)\sin 2\pi\alpha x, \tag{48}$$

where the amplitudes A, B, C, D and E are generally functions of time and are to be determined by the dynamics of the system. Substituting (46) to (47) into (4) to (8) and

equating like terms, we get

$$\dot{A} = \frac{\pi\alpha Q(3\alpha^2-1)}{\alpha^2+1} DE - \mathrm{Pr}\left(\frac{1}{P_l} + \pi^2(\alpha^2+1)\right)A - \pi QD - \frac{\alpha\,\mathrm{Pr}\,\mathrm{Ra}\,B}{\pi(\alpha^2+1)}, \tag{49}$$

$$\dot{B} = -\pi^2(\alpha^2+1)B - \pi\alpha A - \pi^2\alpha AC, \tag{50}$$

$$\dot{C} = \frac{\pi^2}{2}\alpha AB - 4\pi^2 C, \tag{51}$$

$$\dot{D} = \pi^2\alpha AE + \pi A - \frac{\pi^2}{S}(\alpha^2+1)D, \tag{52}$$

$$\dot{E} = -\frac{\pi^2}{2}\alpha AD - \frac{4\pi^2\alpha^2}{S}E, \tag{53}$$

where the dot corresponds to a time derivative.

This set of non-linear ordinary differential equations is not amenable to analytical treatment for the general time-dependent variables and we have to solve it using a numerical method. However, in the case of steady motions, these equations can be solved analytically. Such solutions are very useful because they show that a finite amplitude steady solution to the system is possible for subcritical values of the Rayleigh number and that the minimum values of Ra for which steady solution is possible lies below the critical values for instability to either a steady infinitesimal disturbance or an overstable infinitesimal disturbance.

Thus, if the system is steady, (49) to (53) take the form

$$\pi^2(\alpha^2+1)\left((1/P_l)+\pi^2(\alpha^2+1)\right)A + \pi\alpha\mathrm{Ra}B + (\pi^3/\mathrm{Pr})Q(\alpha^2+1)D - \frac{\pi^4\alpha Q(3\alpha^2-1)}{\mathrm{Pr}}DE = 0, \tag{54}$$

$$\pi^2\alpha AC + \pi\alpha A + \pi^2(\alpha^2+1)B = 0, \tag{55}$$

$$\frac{\pi^2}{2}\alpha AB - 4\pi^2 C = 0, \tag{56}$$

$$\pi^2\alpha AE + \pi A - \frac{\pi^2}{S}(\alpha^2+1)D = 0, \tag{57}$$

$$\frac{\pi^2}{2}\alpha AD + 4\pi^2\frac{\alpha^2}{S}E = 0. \tag{58}$$

Equations (55) to (58) can be rewritten in the form

$$B = -\frac{\pi\alpha A}{\pi^2(\alpha^2+1+(1/8)\alpha^2A^2)}, \tag{59}$$

$$C = (1/8)\alpha AB, \tag{60}$$

$$D = \frac{\pi SA}{\pi^2(\alpha^2+1+(1/8)S^2A^2)}, \tag{61}$$

$$E = -(SAD/8\alpha). \tag{62}$$

Substituting (59) to (62) into (54) and after some simplification, we get

$$\begin{aligned}
&\pi^2 S^2(\alpha^2+1)((1/P_l)+\pi^2(\alpha^2+1))(A^2/8)^3 \\
&+[(\pi^2 S^2(\alpha^2+1)^2(1+(2\alpha^2/S^2)))((1/P_l)+\pi^2(\alpha^2+1)) \\
&-\alpha^2 S^2 \mathrm{Ra}+4\pi^2\alpha^4 Q](A^2/8)^2+[(\pi^2(\alpha^2+1)^3(2+(\alpha^2/S^2)))\times \\
&\times((1/P_l)+\pi^2(\alpha^2+1))-2\alpha^2\mathrm{Ra}(\alpha^2+1)+Q\pi^2\alpha^2(\alpha^2+1)\times \\
&\times(4+((\alpha^2+1)/S^2))](A^2/8)+(\alpha^2(\alpha^2+1)^2/S^2)(\mathrm{Ra}_c^m-\mathrm{Ra}).
\end{aligned} \tag{63}$$

We have to look for real and positive roots of this cubic equation in $A^2/8$, otherwise, the amplitude of the stream function becomes imaginary. So we solve this equation numerically to find the real and positive roots.

4.1 *Finite amplitude analysis with small induced magnetic field and small induced current*

Rudraiah and Vortmeyer [14] have studied the finite amplitude analysis using a Darcy model by setting a constant term in (63) to zero. We note that setting the constant term in (63) equal to zero, is equivalent to restricting the value of Ra near to Ra_c^m. However, this can be avoided by assuming that the terms involving induced magnetic field and induced current are small compared to the applied magnetic field; which is usually the case in many practical problems. In this case (12) takes the form

$$\frac{\partial\eta}{\partial t}+(\mathbf{q}\cdot\nabla)\eta=\frac{\mu_m H_0}{\rho_0}\frac{\partial\xi_0}{\partial z}-\alpha g\frac{\partial T}{\partial x}-\frac{\nu}{k_p}\eta+\nu\nabla^2\eta, \tag{64}$$

where $\eta=\nabla^2\psi$ and $\xi=\nabla^2\phi$,

and all other equations remain the same. Now, substituting (46) to (48) into these equations and assuming, as before, the steady case we get

$$\pi^2(\alpha^2+1)((1/P_l)+\pi^2(\alpha^2+1))A+\pi\alpha\mathrm{Ra}B+(\pi^3 Q/S)(\alpha^2+1)D=0 \tag{65}$$

and the other equations are the same as (55) to (58). As before, substituting (59) to (61) into (65) we get

$$\begin{aligned}
&A[(\pi^2 S^2\alpha^2(\alpha^2+1)((1/P_l)+\pi^2(\alpha^2+1)))(A^2/8)^2 \\
&+(\pi^2 S^2(\alpha^2+1)^2((1/P_l)+\pi^2(\alpha^2+1))+\alpha^4\mathrm{Ra}_c^m-\alpha^2 S^2\mathrm{Ra})(A^2/8) \\
&+\alpha^2(\alpha^2+1)(\mathrm{Ra}_c^m-\mathrm{Ra})]=0.
\end{aligned} \tag{66}$$

The solution $A=0$ corresponds to the pure conduction and the other solutions are given by

$$\begin{aligned}
A^2/8=&[1/((1/P_l)+\pi^2(\alpha^2+1))2\pi^2\alpha^2 S^2(\alpha^2+1)] \\
&\times[\alpha^2 S^2(\mathrm{Ra}-\mathrm{Ra}_c^m)-\alpha^4\mathrm{Ra}_c^m+\pi^2 S^2(\alpha^2+1)Q \\
&\pm((\alpha^2 S^2(\mathrm{Ra}-\mathrm{Ra}_c^m-\mathrm{Ra}_c^m)-\alpha^4\mathrm{Ra}_c^m+\pi^2 S^2(\alpha^2+1)Q)^2 \\
&+4S^2\alpha^4(\alpha^2\mathrm{Ra}_c^m-\pi^2(\alpha^2+1)Q))^{1/2}].
\end{aligned} \tag{67}$$

To ensure the amplitude of the stream function to be real, we have to take the positive sign in front of the radical in (67).

Consider the case where finite solutions exist for $\mathrm{Ra} < \mathrm{Ra}_c^m$. We know that (Veronis [19]) the minimum value of Ra for which solutions exist is that value of Ra which makes the discriminant zero provided that

$$\mathrm{Ra}_f = \frac{\pi^2(\alpha^2+1)}{S^2}\left[\left\{\frac{(S^2-\alpha^2)(\alpha^2+1)}{\alpha^2}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right)\right\}^{1/2}+Q^{1/2}\right]^2, \quad (68)$$

where Ra_f denotes the Rayleigh number for finite amplitude motion. With this value of Ra_f the amplitudes are real provided that the first term on the right side of (68) be non-negative, or equivalently

$$Q > \frac{\alpha^2(\alpha^2+1)}{(S^2-\alpha^2)}\left(\frac{1}{P_l}+\pi^2(\alpha^2+1)\right). \quad (69)$$

Conditions (68) and (69) are meaningful only when

$$S > \alpha. \quad (70)$$

We note that the conditions (69) and (70) are readily accessible in a laboratory experiment with fibre material for porous media and with mercury flowing through it.

We note that the analysis of Brinkman model given above reduces to the Darcy case (Rudraiah and Vortmeyer [14]) if $\pi^2(\alpha^2+1) \ll 1/P_l$ *i.e.*, neglecting $\pi^2(\alpha^2+1)$ compare to $1/P_l$ in $(1/P_l)+\pi^2(\alpha^2+1)$. Further it reduces to the magnetoconvection in the absence of a porous medium if $1/P_l \ll \pi^2(\alpha^2+1)$ *i.e.*, neglecting $1/P_l$ compare to $\pi^2(\alpha^2+1)$ in $((1/P_l)+\pi^2(\alpha^2+1))$. This is the same as the limiting case $P_l \to \infty$.

This shows that the Brinkman model generalises the problem in the sense that with a suitable limit on P_l, we can obtain the results for Darcy flow and pure viscous flow.

5. Heat transport

In the study of convection problems the determination of heat transport across the layer plays a very important role. This is becuase the onset of convection as the Rayleigh number is increased is more readily detected by its effect on the heat transfer. In the quiescent state, the heat transfer is usually due to conduction (radiative heat transfer is usually neglected). Hence if H_t is the rate of heat transfer per unit area,

$$H_t = -K_d \left\langle \frac{\partial}{\partial z} T_{\text{total}} \right\rangle_{z=0} \quad (71)$$

where the angular brackets $\langle \text{---} \rangle$ correspond to a horizontal average with the definition of T_{total} defined earlier. Equation (71) can be written in the form

$$H_t = \frac{\Delta T}{d} - \frac{\Delta T}{d} \sum_{n=1}^{N} n\pi b_{\text{on}}, \quad (72)$$

with the restriction $N = 2$. The second term on the right side of (72) represents that the heat which enters at the bottom by conduction is carried on to the top by both conduction and convection and hence the heat transfer increases above that given by conduction alone. This process can be explained physically by the relationship between the driving temperature difference ΔT and the heat transport. In dimensionless variables, this is the Rayleigh-Nusselt-number curve. Thus from (72), the Nusselt

number is

$$Nu = \frac{H_t d}{K_d \Delta T} = 1 - \sum_{n=1}^{N} n\pi b_{on} = 1 - 2\pi C \tag{73}$$

with $N = 2$, where C is given by (60). For Rayleigh numbers below the critical value, the heat transport is purely by conduction for $A = 0$ and B, C, D and E are all zero, where B, C, D and E are given by (59) to (62). In that case (73) shows Nu to be unity.

6. Numerical experiments

The minimum values of Ra_c^m, Ra_c^0, Ra_f and their corresponding values of α^2 for given values of Q, P_l, S are calculated by small variation of α^2. Ra_c^0 is calculated with the condition (24) on Q. The results of these calculations are tabulated in table 1 for $P_l = 10^{-3}$ and $S = 4$ and 8. The typical behaviour is observed for other values of P_l.

To evaluate heat transport, we have to solve the cubic equation (63) for $(A^2/8)$. Here we solve the cubic equation (63) by using a system subroutine POLRT of DEC 1090 system and we calculate Nusselt number from (73). The results of heat transport are tabulated in table 2 and the results are discussed in §6.

7. Discussion

Magnetoconvection through a sparsely packed porous medium has been investigated in detail using linear stability analysis and finite amplitude analysis with limited representation. The following conclusions are drawn from the present investigation:

Table 1. Values of Ra_c^m, Ra_c^0, Ra_f and their corresponding values of α^2 for $P_l = 10^{-3}$

S	Q	Ra_c^m ($\times 10^5$)	α^2	Ra_c^0 ($\times 10^5$)	α^2	Ra_f ($\times 10^5$)	α^2
4	10^{-3}	0·402540200	0·980	—	—	0·37615010	1·140
	10^{-2}	0·402542000	0·980	—	—	0·37645490	1·140
	10^{-1}	0·402559990	0·980	—	—	0·37741950	1·140
	10^{0}	0·402739400	0·980	—	—	0·38047490	1·130
	10^{1}	0·404530400	0·985	—	—	0·39019010	1·115
	10^{2}	0·422233500	1·030	—	—	0·42145590	1·065
	10^{3}	0·584842300	1·380	0·55369300	1·070	0·52573740	0·935
	10^{4}	0·186197800 × 10^6	3·175	0·70869840	1·415	0·67628390	0·705
8	10^{-3}	0·402540200	0·980	—	—	0·39634190	1·015
	10^{-2}	0·402542000	0·980	—	—	0·39649360	1·010
	10^{-1}	0·402560000	0·980	—	—	0·39697310	1·010
	10^{0}	0·402739400	0·980	—	—	0·39849160	1·010
	10^{1}	0·404530400	0·985	—	—	0·40330850	1·005
	10^{2}	0·422233500	1·030	—	—	0·41868180	0·985
	10^{3}	0·584842300	1·380	0·47814680	1·030	0·46870360	0·930
	10^{4}	0·186197800 × 10^6	3·175	0·67628385	1·415	0·6406941	0·800

Table 2. The values of Nu and Ra for different values of Q and S for $P_l = 10^{-3}$.

S	$Q = 10^{-3}$		$Q = 10$		$Q = 10^3$		$Q = 10^4$	
	Nu	Ra	Nu	Ra	Nu	Ra	Nu	Ra
4	1·012145	$4·050 \times 10^4$	1·002325	$4·05 \times 10^4$	1·000261	$5·850 \times 10^4$	1·001866	$1·870 \times 10^5$
	1·385823	$5·000 \times 10^4$	1·383947	$5·00 \times 10^4$	1·047116	$6·000 \times 10^4$	1·041712	$2·010 \times 10^5$
	1·651178	$6·000 \times 10^4$	1·754479	$6·50 \times 10^4$	1·295633	$6·025 \times 10^4$	2·193092	$2·020 \times 10^5$
	1·841486	$7·000 \times 10^4$	2·187899	$1·00 \times 10^5$	1·626226	$6·500 \times 10^4$	2·474843	$2·150 \times 10^5$
	1·984704	$8·000 \times 10^4$	2·673922	$2·50 \times 10^5$	1·794893	$7·000 \times 10^4$	2·588647	$2·300 \times 10^5$
	2·672601	$2·500 \times 10^5$	2·863926	$6·00 \times 10^5$	2·158849	$9·000 \times 10^4$	2·738430	$2·750 \times 10^5$
	2·882796	$7·000 \times 10^5$	2·897917	$8·00 \times 10^5$	2.546314	$1·500 \times 10^5$	2·958995	$1·000 \times 10^6$
	2·917925	$1·000 \times 10^6$	2·918320	$1·00 \times 10^6$	2·939187	$1·005 \times 10^6$	2·993943	$6·000 \times 10^6$
8	1·126121	$4·300 \times 10^4$	1·002379	$4·05 \times 10^4$	1·308728	$5·200 \times 10^4$	1·980130	$1·138 \times 10^5$
	1·474303	$5·300 \times 10^4$	1·384743	$5·00 \times 10^4$	1·499350	$5·500 \times 10^4$	2·101557	$1·151 \times 10^5$
	2·020217	$8·300 \times 10^4$	2·187705	$1·00 \times 10^5$	2·022799	$7·500 \times 10^4$	2·690284	$1·800 \times 10^5$
	2·208917	$1·030 \times 10^5$	2·836736	$5·00 \times 10^5$	2·629199	$1·750 \times 10^5$	2·809022	$2·500 \times 10^5$
	2·767834	$3·530 \times 10^5$	2·883341	$7·00 \times 10^5$	2·864103	$4·550 \times 10^5$	2·933862	$6·000 \times 10^5$
	2·909110	$9·030 \times 10^5$	2·909247	$9·00 \times 10^5$	2·924147	$8·050 \times 10^5$	2·962097	$1·000 \times 10^6$
	2·919762	$1·023 \times 10^6$	2·918317	$1·00 \times 10^6$	2·939438	$1·005 \times 10^6$	2·994878	$7·000 \times 10^6$

From the linear analysis we find that oscillatory (overstable) motions occur for restricted range of S (> 1 which is the ratio of thermal diffusivity to magnetic diffusivity) and Q for given value of the porous parameter P_l, because they can reduce the stabilizing effect of Lorentz force. For given P_l the effect of strong magnetic field is to suppress convection—direct (steady) or oscillatory (overstable) depending on the values of S and Q. From table 1, it is clear that convection sets in as a direct (or marginal) mode at the critical Rayleigh number Ra_c^m when $S < 1$ and it may first appear as an oscillatory mode at $\mathrm{Ra} = \mathrm{Ra}_c^0$ when $S > 1$ in the absence of porous media (see [2, 3] and [20]). In the presence of a porous medium, the same phenomena is also true. Also, the porous medium stabilizes the system to the maximum extent when the usual viscous dissipation is also present in addition to Darcy's resistance. The physical reasons for this is that, the energy released by buoyancy force acting on the fluid must balance the energy released by Darcy's resistance, Joule heating, and the usual viscous dissipation. This can be achieved only at higher temperature gradients.

In general we conclude that for smaller values of Q the motion should be steady, infinitesimal or finite, rather than overstable motion. This should be comparable to that which occurs for hydrodynamic convection through a porous medium (Lapwood [6]). We find that the generation of zonal current by the magnetic field is responsible for oscillatory motions which exist at Rayleigh number smaller than that of infinitesimal steady motions. This zonal current is also responsible for the existence of finite amplitude motions.

Therefore, finite amplitude steady convection is investigated using Veronis's [19] truncated representation. We find that for large values of Q finite amplitude steady convections are not possible because they occur only in a subcritical range of Ra as long as they can reduce the constraint of the magnetic field. This is because, greater amplitudes require more release of potential energy which in turn requires a larger value of Ra. It is also found that, no finite amplitude motions are possible when the

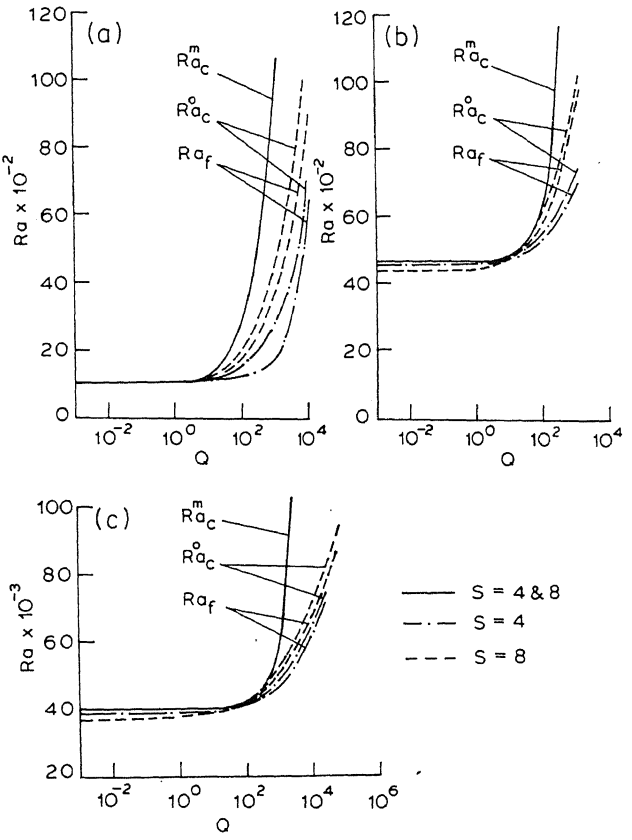

Figure 3. Curves of Ra_c^m, Ra_c^0 and Ra_f as function of Q for $P_l = 10^{-1}$, 10^{-2} and 10^{-3} (Brinkman model).

parameter $S > \alpha$, the preferred scale of finite amplitude motions is larger than the corresponding scale for infinitesimal oscillatory motions.

The existence of finite amplitude steady convection for subcritical values of Ra for different values of Q, S and P_l are shown in figures 3a–c.

From these figures, we conclude that, the finite amplitude steady convection exists for subcritical values of Ra for all $Q > 7$ for $P_l \leqslant 10^{-1}$. However for $P_l = 10^{-3}$ the finite amplitude steady convection exists for subcritical values of Ra for all Q and these values of Ra for finite amplitude steady convection are small in the case of $S = 4$ compared to that of $S = 8$.

The heat transport given by (73) is numerically evaluated for different values of Ra, Q, S and α^2 and the results are tabulated in table 2 and depicted in figures 4 to 6 and we found that:

(a) For a fixed value of Q, Nu increases with S (for $S = 4$ and 8) which means that a fluid having less magnetic diffusivity transports more heat for a fixed value of Q.

(b) As the value of P_l is decreased, instability sets in at a higher value of Rayleigh number as in the rotating single and two component systems (Rudraiah and Srimani [15]; Srimani [16]). We find that chances of subcritical instabilities occurring are less for smaller values of P_l ($< 10^{-3}$). This is because no fluid exhibits subcritical instability

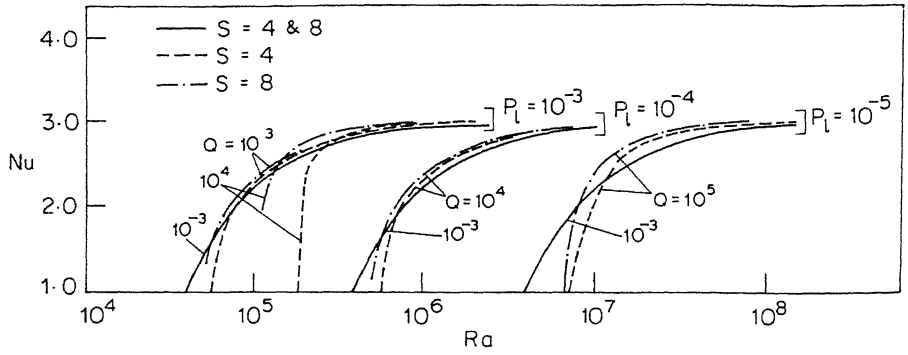

Figure 4. Curves of heat transport Nusselt number (Nu) against Rayleigh number (Ra) for (i) $P_l = 10^{-3}$ (ii) $P_l = 10^{-4}$ and (iii) $P_l = 10^{-5}$.

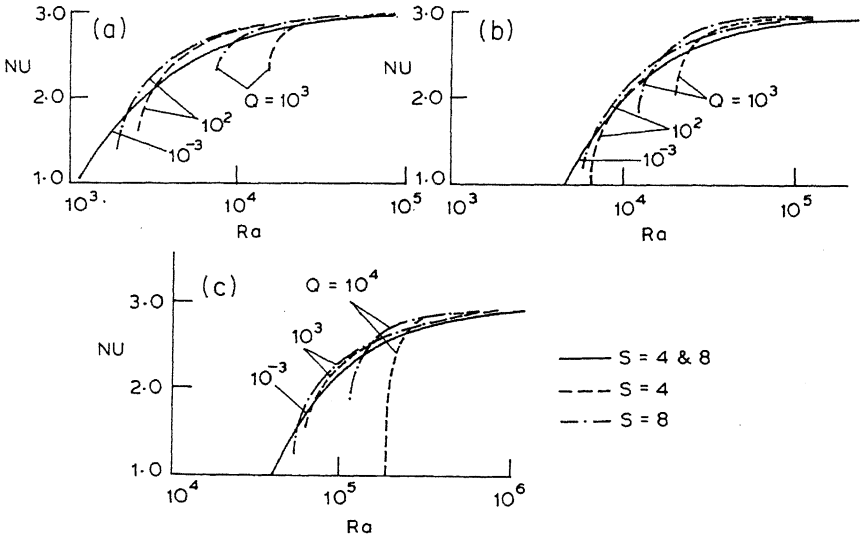

Figure 5. Curves of heat transport Nusselt number (Nu) against Rayleigh number (Ra) for Brinkman model (**a**) $P_l = 10^{-1}$ (**b**) $P_l = 10^{-2}$ and (**c**) $P_l = 10^{-3}$.

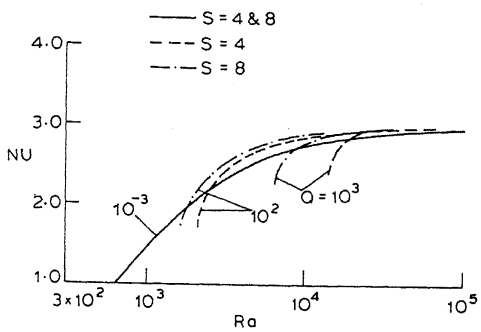

Figure 6. Curves of heat transport Nusselt number (Nu) against Rayleigh number (Ra) for viscous case $P_l = 10^3$.

for $P_l = 10^{-5}$ (see figure 4) in the range $10^{-3} \leqslant Q \leqslant 10^{-5}$. Whereas for $P_l = 10^{-4}$ and 10^{-3}, fluids having $S = 8$ exhibits subcritical instability but not for $S = 4$ (figures 4–6). This means that as P_l increases the system becomes more and more stable to finite amplitude perturbations as the magnetic diffusivity is decreased.

In this context we can note that some fluids ($2 \leqslant S \leqslant 8$) do exhibit subcritical instabilities for $P_l = 10^{-1}$ and 10^{-2} (figures 5a, b). Whereas, for $P_l = 10^{-3}$ instabilities exist only for $S = 8$ (figure 5c) which is similar to the values of $P_l > 10^{-3}$.

In the viscous case ($P_l \to \infty$) all the fluids considered exhibit subcritical instabilities as expected (Weiss [21, 22]). Further, for $Q \leqslant 10^{-3}$ we note that subcritical instabilities do not exist and there is no difference in the values of Nu for $S = 4$ and 8 (figures 4 to 6). We also note that Nu increases with P_l and the values of the heat transport in the case of a porous medium are less compared to the pure viscous case ($P_l \to \infty$) (see figure 6).

For values of $P_l = 10^{-1}$ and 10^{-2}, heat transport decreases with increasing Q for the fluid $S = 4$, whereas it increases with increasing Q for $S = 8$. But for the case $P_l = 10^{-3}$, heat transport decreases with increasing Q for $S = 4$, while it increases with increasing Q for $S = 8$ (figure 5).

Acknowledgement

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Principal bundles on the affine line

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Abstract. We prove that any principal bundle on the affine line over a perfect field with a reductive group as structure group comes from the base field by base change.

Keywords. Affine line; reductive algebraic group; principal bundle.

1. Introduction

Let k be an arbitrary field and G a linear algebraic group over k. Let $\mathbf{A}$ denote the affine line Spec $k[T]$, where $k[T]$ is the polynominal ring in one variable over k. We prove the following result in this paper.

THEOREM 1.1. Assume that G is connected and reductive. Let $B \to \mathbf{A}$ be a principal G-bundle on $\mathbf{A}$ such that $B \underset{\text{Spec } k}{\times} \text{Spec } k_s$ is trivial over Spec $k_s[T]$ where k_s is the separable closure of k. Then there is a principal G-bundle B_0 over Spec k such that B is k-isomorphic to the pull back of B_0 by the structure morphism $\mathbf{A} \to \text{Spec } k$.

If k is algebraically closed it follows from a theorem of Steinberg [16] that all G-bundles on $\mathbf{A}$, with G connected, are trivial (cf. [13]). Thus if k is perfect (in particular if the characteristic of k is zero) the assumption of triviality over k_s is always satisfied. On the other hand Knus *et al* [11] have shown that for separably closed non-perfect fields k there exist non-trivial $PGL(n)$-bundles on $\mathbf{A}$. However it seems likely that for simply connected groups G such a phenomenon does not occur.

If the base field k is of positive characteristic the connectedness assumption on G is essential even if k is algebraically closed, for Artin-Schrier extensions of $k(T)$ provide non-trivial finite Galois coverings of $\mathbf{A}$ [12, III §4].

Thus all the assumptions made in the theorem are essential.

The situation regarding non-reductive groups seems rather complicated and we make no efforts to examine it here. The result for many special cases is known: (i) for $GL(n)$ and inner forms of $GL(n)$ it is a reinterpretation of the fact that projective modules over $D[X]$, D a division algebra, are free (ii) for $G = Sp(n)$ it is the same as the classification of non-degenerate alternating forms over $k[X]$ and (iii) for $G = SO(n)$, char $k \neq 2$, it is due to Harder (see [10]). He used the method of extending from $\mathbf{A}$ to the projective line $\mathbf{P}$ which idea we have followed here.

2. Preliminary reductions

Throughout this paper k will denote an arbitrary field, k_s its separable closure and $\bar{k}$ its algebraic closure. For any scheme X over Spec k and a k-algebra R we denote the base

change $X \underset{k}{\times} \operatorname{Spec} R$ by X_R and the R-valued points of X, *i.e.* the set of k-morphisms $\operatorname{Spec} R \to X$, by $X(R)$.

2.1 By a (principal) G-bundle over X we mean a scheme $E \to X$ over X together with an action of G (on the right) on E such that the morphism

$$E \underset{k}{\times} G \to E \underset{X}{\times} E$$

given by $(e, g) \to (e, eg)$ where e, g are R-valued points of E, G respectively, is an isomorphism (cf. [12], Chapter III §4 and [14]).

2.2 It is known ([14] or [12] Chapter III §4.2) that for a smooth G any G-bundle is locally trivial in the étale topology, *i.e.* X is covered by étale morphisms $f: U \to X$ such that f^*E is isomorphic to the trivial bundle $U \times G$ on U.

2.3 A section $\sigma: X \to E$ canonically gives rise to a trivialisation $\tilde{\sigma}: X \times G \to E$ defined by $\tilde{\sigma}(x, g) = \sigma(x)g$ where x, g are R-valued points of X, G and $\sigma(x)g$ is the translate of $\sigma(x)$ by g given by the action of G on X.

2.4 If F is a quasi-projective scheme and the smooth group G acts on F we can then form, using étale descent, the associated bundle $E(F) \to X$ with fibre F. Recall, [14], that $E(F)$ is the quotient of $E \underset{k}{\times} F$ by equivalence $(e, f) \sim (eg, g^{-1}f)$ where e, f, g are R-valued points of E, F, G respectively.

2.5 A section $\sigma: X \to E$ also gives a trivialisation $X \times F \to E(F)$ of any associated bundle $E(F)$ defined by sending (x, f) to the equivalence class of $(\sigma(x), f)$ in $E \times F$ where x, e, f are R-valued points X, E, F. We call an isomorphism $X \times F \to E(F)$ an *allowable trivialisation* if it comes from a section of E as above.

2.6 Let π be the Galois group of k_s over k. Then a standard argument ([12], Ex. 2.6 p. 93 and Chapter III §4) shows that the isomorphism classes of G-bundles on the affine line $\mathbf{A} = \operatorname{Spec} k[T]$ which become trivial on $\mathbf{A}_{k_s}$ are in natural one to one correspondence with the Galois cohomology set $H^1(\pi, G(k_s[T]))$, (see [15]). Thus our theorem can be interpreted as saying that the natural map $H^1(\pi, G(k_s)) \to H^1(\pi, G(k_s[T]))$ is a bijection when G is a connected reductive group. (The injectivity of this map is clear since it has a section given by restriction to a k-rational point).

PROPOSITION 2.7. Let S be a torus over k. Then any S-bundle on $\mathbf{A}$ becomes trivial on $\mathbf{A}_{k_s}$ and is obtained from an S-bundle on $\operatorname{Spec} k$ by base change.

Proof. The torus S_{k_s} over k_s splits into a product of, say n copies of, the multiplicative group G_m. Now a G_m-bundle is equivalent to a line bundle or again a projective module of rank 1 over $k[T]$. Since $k[T]$ is a principal ideal domain any projective module is trivial. Hence the first assertion. To see that the S-bundle comes from k we observe that $S(k_s[T]) = S_{k_s}(k_s[T]) = G_m(k_s[T])^n = S(k_s)$. Therefore $H^1(\pi, S(k_s[T])) = H^1(\pi, S(k_s))$.

Suppose now G is a connected reductive group and $B \to \mathbf{A}$ a G-bundle. Let $p: \operatorname{Spec} k \to \mathbf{A}$ be a k-rational point of $\mathbf{A}$. Then the pull back p^*B is a G-bundle on $\operatorname{Spec} k$. Twisting G by this principal homogeneous space, *i.e.* forming the associated bundle of p^*B for the adjoint action of G on itself ([15] I §5.3) we obtain a new connected

reductive k-group G', a k-form of G. We can also twist the bundle B by the pull back of p^*B by the structure morphism $\mathbf{A} \to \operatorname{Spec} k$ thereby obtaining a G'-bundle B' on $\mathbf{A}$. It is easy to see that the restriction of B' to the point p is trivial and that B comes from Spec k if and only if B' does. Thus we have proved the following claim.

CLAIM 2.8. To prove Theorem 1.1 we can assume without loss of generality that the bundle B is trivial when restricted to a given k-rational point of $\mathbf{A}$.

Next let G, $B \to \mathbf{A}$ be as above and $H = G/[G, G]$. Let $\varphi: G \to H$ be the natural projection. We then obtain from B by extension of structure group a H-bundle $B_H \to \mathbf{A}$ which is trivial over a k-rational point of $\mathbf{A}$. Since the theorem holds for tori (Proposition 2.7) and H is a torus, B_H is trivial. This means that B admits a reduction of structure group to $[G, G]$ (cf. [14], [13]). This proves the following claim.

CLAIM 2.9. It is enough to prove theorem 1.1 for G-bundles B such that G is semisimple and B restricted to a k-rational point of $\mathbf{A}$ is trivial.

Our method of proof of the theorem will be to show that G-bundles satisfying the assumptions of the above claim and the theorem admit a reduction of structure group to a suitable maximal torus T of G. Then we can appeal to Proposition 2.7 to conclude the theorem. For this purpose we will need the following lemma (2.10 below).

Let $\underline{G} \to X$ be a semisimple group scheme over X. A subgroup scheme $\underline{S} \subset \underline{G}$ is called a *maximal torus subgroup scheme* (or a *maximal subtorus*) if for any $x \in X(\bar{k})$ the fibre $\underline{S}_x$ is a maximal torus of the fibre $\underline{G}_x$ of $\underline{G}$.

Let G be a semisimple group over k and $\tilde{G} \to G$ a covering group (a central isogeny). Let $E \to X$ be a G-bundle. Let T be a maximal torus of G and $\tilde{T}$ the maximal torus of $\tilde{G}$ which is the inverse image of T. Let $E(\tilde{G}) \to X$ be the group scheme given by the associated bundle with fibre $\tilde{G}$ for the adjoint action of G on $\tilde{G}$.

LEMMA 2.10. With the above notation suppose that there is an embedding $\varphi: X \underset{k}{\times} \tilde{T} \to E(\tilde{G})$ of the constant torus scheme $X \underset{k}{\times} \tilde{T}$ as a maximal sub-torus of $E(\tilde{G})$ with property that X is covered by étale morphisms $f: U \to X$ such that there is an allowable trivialisation (see §2.5) $\psi: U \underset{k}{\times} \tilde{G} \to f^*E(\tilde{G})$ whose restriction to $U \underset{k}{\times} \tilde{T}$ coincides with $f^*\varphi$. Then φ gives rise to a natural reduction of structure group of E to T.

Proof. Let $f_i: U_i \to X$ be a covering of X by étale morphisms and $\sigma_i: U_i \to f_i^*E$ be sections giving allowable trivialisations, again denoted by σ_i, $\sigma_i: U_i \times \tilde{G} \to f_i E(\tilde{G})$ extending $f_i^*\varphi$. Let $\bar{\sigma}_i: U_i \to f_i^*E(G/T)$ be the composite of σ_i with the projection $f_i^*E(G) \to f_i(E(G/T))$. Since any inner automorphism of $\tilde{G}_{k_s}$ which is identity on $\tilde{T}$ comes from T_{k_s} it follows that in the fibre product $U_i \underset{X}{\times} U_j$ the pull backs of $\bar{\sigma}_i$ and $\bar{\sigma}_j$ coincide locally in the étale topology and hence by étale descent on the whole of $U_i \underset{X}{\times} U_j$. Again by étale descent the $\bar{\sigma}_i$ patch up to give a section $\bar{\sigma}: X \to E(G/T) = E/T$ which is the required reduction of structure group to T.

DEFINITION 2.11. Let X be a k-scheme and $\overline{X} = X_{\bar{k}}$. Let $J \subset \mathcal{O}_{\overline{X}}$ be the ideal sheaf of

nilpotent elements. Let $I \subset \mathcal{O}_X$ be the intersection of all the ideal sheaves of $\mathcal{O}_X$ whose extensions to $\mathcal{O}_{\bar{X}}$ contain J. We then call the closed k-subscheme X_0 of X corresponding to I the *separably reduced* of X. (Leema 2.13 below offers some justification for this terminology).

Remarks 2.12. Note that the extension of I to $\mathcal{O}_{\bar{X}}$ contains J and hence X_0 is an absolutely reduced scheme. Further if Y is an absolutely reduced scheme then any k-morphism $Y \to X$ factors through X_0. If the base field k is perfect any reduced scheme is absolutely reduced and the separably reduced of X coincides with the reduced of X.

LEMMA 2.13. With the above notation, $X_0(k_s) = X(k_s)$ and X_0 is characterised by the property that $X_0\ (\bar{k})$ is the Zariski closure of $X(k_s)$ in $X(\bar{k})$.

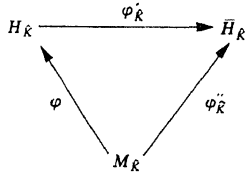
Proof. This follows from the fact that a reduced k-scheme Y is absolutely reduced if and only if $Y(k_s)$ is dense in $Y(\bar{k})$ (see [1], theorem 14.4 Chap. AG).

3. Construction of a group scheme over P

Let G be a connected semisimple group over k and $\tilde{G}$ its simply connected covering. Let $B \to \mathbf{A}$ be a G-bundle which becomes trivial on $\mathbf{A}_{k_s}$. Let $H = E(\tilde{G}) \to \mathbf{A}$ be the associated group scheme for the adjoint action of G on $\tilde{G}$. We will now extend the group scheme H to a smooth group scheme $\bar{H} \to \mathbf{P}$ over the projective line $\mathbf{P} = \mathbf{A} \cup \{\infty\}$, though $\bar{H}_\infty$ the fibre at ∞ may not be reductive.

Let R be the local ring of $\mathbf{P}$ at the point ∞ and $\hat{R}$ its completion with respect to the maximal ideal. Let K be the quotient field of R and $\hat{K}$ its completion (with respect to the valuation corresponding to ∞). Note that $\hat{K}$ is the quotient field of $\hat{R}$.

LEMMA 3.1. Suppose $M \to \operatorname{Spec} \hat{R}$ is a smooth affine group scheme and $\varphi : M_{\hat{K}} \to H_{\hat{K}}$ an isomorphism of $\hat{K}$-groups where $M_{\hat{K}}$ is the base change of M by $\hat{R} \to \hat{K}$ and $H_{\hat{K}}$ that of H by $R \subsetneq K \subsetneq \hat{K}$. Then there is a smooth group scheme $\bar{H} \to \mathbf{P}$, such that there are isomorphisms of group scheme $\varphi' : H \to \bar{H}|\mathbf{A}$ and $\varphi'' : M \to \bar{H}_{\hat{R}}$ making the diagram commutative.



Proof. Consider the morphisms $\mathbf{A} \subsetneq \mathbf{P}$ and $\operatorname{Spec} \hat{R} \to \mathbf{P}$. Since $\hat{R}$ is flat over R these two morphisms give a covering of $\mathbf{P}$ in the faithfully flat quasi-compact topology. The fibre product $\mathbf{A} \underset{\mathbf{P}}{\times} \operatorname{Spec} \hat{R}$ is $\operatorname{Spec} K$. Since $M \to \mathbf{A}$ and $N \to \operatorname{Spec} \hat{R}$ are affine the lemma is an immediate consequence of faithfully flat descent ([12] I, theorem 2.23 p. 19).

PROPOSITION 3.12. There is a smooth affine group scheme $\bar{H} \to \mathbf{P}$ extending $H \to \mathbf{A}$ with the following properties.

(a) There is a finite Galois extension L of k such that the base change $\bar{H}_L \to \mathbf{P}_L$ admits a constant torus subgroup scheme $\varphi : \mathbf{P}_L \underset{L}{\times} S \to \bar{H}_L$ where S is a maximal torus of $\tilde{G}_L$ which is split.

(b) On $\mathbf{A}_L$ there is an allowable trivialisation (see §2.5) $\mathbf{A}_L \underset{k}{\times} \tilde{G} \to H_L$ extending $\varphi|_{\mathbf{A}_L \underset{L}{\times} S}$.

The rest of this section will be devoted to proving this proposition. We will obtain the extension $\bar{H}_L$ by making use of the theory of Bruhat-Tits for groups over local fields ([5] §5, also [3], [4]). To get the constant subtorus we will also make use of the theorem of Grothendieck-Harder ([6], [9], [13]) on G-bundles over $\mathbf{P}$.

By assumption $B_{k_s} \to \mathbf{A}_{k_s}$ is trivial. Since all the schemes involved are of finite type over k we can find a finite Galois extension L over k such that (i) $B_L \to \mathbf{A}_L$ is trivial and (ii) $\tilde{G}$ has a maximal torus S defined and split over L.

Let $R' = R \underset{k}{\otimes} L$ be the local ring of $\mathbf{P}_L$ at ∞ and $K' = K \underset{k}{\otimes} L$ be its quotient field. Let $\hat{R}'$ and $\hat{K}'$ be the respective completions with respect to the valuation corresponding to ∞. Note that $\hat{R}' = \hat{R} \underset{k}{\otimes} L$ and $\hat{K}' = \hat{K} \underset{k}{\otimes} L$ and the Galois group of $\hat{K}'$ over $\hat{K}$ is the same as the Galois group π of L over k.

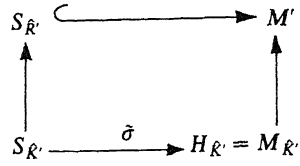
It follows from Lemma 3.1 that to get an extension of H_L to $\mathbf{P}_L$ it is enough to construct a smooth affine group scheme $M \to \operatorname{Spec} \hat{R}'$ together with an isomorphism $\varphi : H_{\hat{K}'} \to M_{\hat{K}'}$ of the generic fibres.

Consider the semisimple simply connected split group $H_{\hat{K}'}$ over the complete local field $\hat{K}'$. Let $P \subset H_{\hat{K}'}(\hat{K}')$ be a parahoric subgroup of $H_{\hat{K}'}$ (see [3] §2). Then according to Bruhat-Tits if A is the subring of the coordinate ring of $H_{\hat{K}'}$ consisting of functions which take values in $\hat{R}'$ on P then $M' = \operatorname{Spec} A \to \operatorname{Spec} \hat{R}'$ is a smooth affine group scheme such that the natural map $A \underset{\hat{R}'}{\otimes} \hat{K}'$ into the coordinate ring of $H_{\hat{K}'}$ is an isomorphism. Thus the generic fibre of M' is canonically isomorphic to $H_{K'}$ and therefore M' gives an extension $\bar{H}' \to \mathbf{P}_L$ of H_L (Lemma 3.1).

If moreover the parahoric subgroup P is invariant under the natural action of the Galois group π on $H_{\hat{K}'}(\hat{K}')$ then π acts on M' compatibly with its action on $H_{\hat{K}'}$. Hence π then acts on $\bar{H}'$. By Galois descent it then follows that M' and $\bar{H}'$ descend to $M \to \operatorname{Spec} \hat{R}$ and $\bar{H} \to \mathbf{P}$ respectively.

The fixed point theorem of Bruhat-Tits ([5], §5) guarantees the existence of a π-invariant parahoric subgroup P_0. Let $H'_0 \to \mathbf{P}_L$ and $\bar{H}_0 \to \mathbf{P}$ be the extensions corresponding to P_0.

Now fix a section $\sigma : \mathbf{A}_L \to B_L$. Then σ gives an allowable trivialisation $\tilde{\sigma} : \mathbf{A}_L \underset{k}{\times} \tilde{G} \to H_L$. Its restriction $\tilde{\sigma}|\mathbf{A}_L \underset{L}{\times} S$ gives a constant subtorus of H_L. Let I be the apartment of the Bruhat-Tits building corresponding to the split torus $\tilde{\sigma}(S_{\hat{K}'})$. Then if P is a parahoric subgroup belonging to I by [4] we have a canonical embedding of the constant torus $S_{\hat{R}'} = \operatorname{Spec} \hat{R}' \underset{L}{\times} S$ in M' such that the diagram below commutes.



As in 3.1 this implies that the subtorus $\tilde{\sigma}$: $\mathbf{A}_L \underset{L}{\times} S \to H_L$ extends to φ: $\mathbf{P}_L \underset{L}{\times} S \to \bar{H}'$.

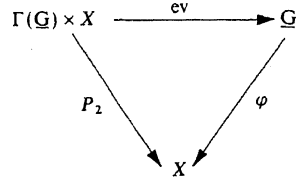
Now by the conjugacy theorem for parahoric subgroups ([3] §3) we can find a parahoric subgroup P_1 belonging to I such that $P_0 = \operatorname{Int} g(P_1)$ where $g \in H_{\hat{K}'}(\hat{K}')$. Since parahoric subgroups are open and $H_{K'}(K')$ is dense in $H_{\hat{K}'}(\hat{K}')$ we can take g in $H_{K'}(K')$. Then there exist sections $\alpha: \mathbf{A}_L \to H_L$, $d \in S_{K'}$ and $\beta \in P_1$ such that $g = \alpha_{K'} \mathrm{d}\beta$. This is essentially a consequence of the theorem of Grothendieck-Harder that any G_L-bundle on $\mathbf{P}_L$ admits a reduction of structure group to a split maximal torus (see [13], theorem 4.2).

Let $P_2 = \operatorname{Int} d(P_1)$. Then P_2 belongs to I and hence we have the constant subtorus $\varphi_2 : \mathbf{P}_L \underset{L}{\times} S \to \bar{H}'_2$ where the latter is the extension of H_L corresponding to P_2. Since $P_0 = \operatorname{Int} \alpha(P_2)$ it is easily seen that $\operatorname{Int} \alpha : H_L \to H_L$ extends to $\psi : \bar{H}'_2 \to \bar{H}'_0$. It now follows that the extension $\bar{H}_0 \to \mathbf{P}$ and the constant subtorus $\varphi = \psi \cdot \varphi_2 : \mathbf{P}_L \underset{L}{\times} S \to \bar{H}'_0$ satisfy the conditions (a) and (b) of the proposition. This completes the proof of Proposition 3.2.

In §4 we shall show how to pass from φ, which is defined only over L, to a subtorus defined over k using the group of global section of $\bar{H}$.

4. Sections of $\bar{H}$

4.1 Let X be a projective scheme over k and $f: \underline{G} \to X$ be a group scheme of finite type, affine over X. Consider the functor which associates to each k-scheme Y the group of global sections of the product group scheme $\varphi \times id : \underline{G} \underset{k}{\times} Y \to X \times Y$. By [9] §1.4 or [7] we know that this functor is representable by a group scheme $\Gamma(\underline{G})$ of finite type over k. We also have the evaluation morphism (the universal global section) ev:



4.2 If $\underline{N} \subset \underline{G}$ is a subgroup scheme then the global section functor of $\underline{N}$ is a subfunctor of that of $\underline{G}$. Therefore $\Gamma(\underline{N})$ is a subgroup scheme of $\Gamma(\underline{G})$.

4.3 If N is an affine k-group scheme then the global sections functor of the constant group scheme $N \underset{k}{\times} X \to X$ is clearly represented by N itself, since X is projective.

4.4 Now we apply the above discussions to the group scheme $\overline{H} \to \mathbf{P}$ constructed in the previous section (Proposition 3.2). Let Γ' be the group of global sections of $\overline{H}$. Let Γ be the separably reduced of Γ' (see §2.11). Since Γ is a group scheme and is absolutely reduced it is a smooth group scheme.

By the rank of a k-group scheme we mean the dimension of a maximal subtorus defined over the algebraic closure $\bar{k}$ of k.

By Proposition 3.2 $\overline{H}$ has the constant maximal torus subgroup scheme $\mathbf{P}_L \underset{L}{\times} \tilde{S}$. Therefore by the remarks in §§4.2 and 4.3 (and §2.12) $\tilde{S}$ is naturally embedded in Γ_L. This shows that rank $\Gamma \geqslant$ rank G. In fact we prove that rank $\Gamma =$ rank G in the proposition below.

PROPOSITION 4.5. $\tilde{S} \subset \Gamma_L$ is a maximal torus of Γ

Proof. Consider the Lie algebra bundle $V \to \mathbf{P}$ of the smooth group scheme $\overline{H} \to \mathbf{P}$. The fibres of V are of constant rank and $V \to \mathbf{P}$ is a vector bundle with Lie algebra structures on fibres. Let $\sigma \in \Gamma(\bar{k})$ be a $\bar{k}$-rational point of Γ. Then σ gives a section $\mathbf{P}_{\bar{k}} \to \overline{H}_{\bar{k}}$ and by the adjoint action gives a section $Ad\,\sigma: \mathbf{P}_{\bar{k}} \to \text{End } V_{\bar{k}}$. Since the ith coefficient of the characteristic polynominal of an endomorphism is invariant under inner conjugation it gives rise to a function $a_i: \text{End } V_{\bar{k}} \to \bar{k}$. Composing with $Ad\,\sigma$ we then get a regular function $a_i\, Ad\,\sigma: \mathbf{P}_{\bar{k}} \to \bar{k}$. This must be a constant since $\mathbf{P}_{\bar{k}}$ is complete. Thus the characteristic polynominal of $Ad\,\sigma(x)$ is the same for all $x \in \mathbf{P}(\bar{k})$.

Now let $S \subset \Gamma_{\bar{k}}$ be a maximal torus of $\Gamma_{\bar{k}}$ and $x \in \mathbf{A}(\bar{k})$. We then have the evaluation morphism $ev_x: S_{\bar{k}} \to \overline{H}_x$. Suppose for some $\sigma \in S(\bar{k})$ we have $ev_x(\sigma) = \sigma(x)$ is the identity. Then $Ad\,\sigma(x) = id.$ for any $y \in \mathbf{P}^1(\bar{k})$, $Ad\,\sigma(y)$ is semisimple since it is the image of the semisimple element $\sigma \in S(\bar{k})$. Further it has the same characteristic polynominal as $Ad\,\sigma(x)$. Therefore it must be the identity. This shows that on $\mathbf{A}$ the section $\sigma: \mathbf{A}_{\bar{k}} \to B_{\bar{k}}(\tilde{G})$ factors through the kernel $B_{\bar{k}}(F) \subset B_{\bar{k}}(\tilde{G})$ where F is the kernel of $\tilde{G} \to G$. Since F is a finite group scheme the reduced of its identity component is the trivial group $\{1\}$. Therefore the reduced of the identity component of $B_{\bar{k}}(F)$ is $B_{\bar{k}}(\{1\})$. Since $\mathbf{A}_{\bar{k}}$ is reduced and connected the morphism $\sigma: \mathbf{A}_{\bar{k}} \to B_{\bar{k}}(F)$ then factors through $B_{\bar{k}}(\{1\})$ which shows that $\sigma = id.$

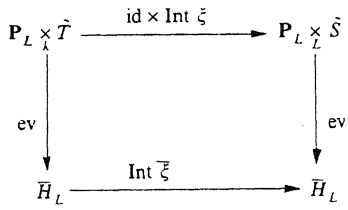
Thus we have proved that $ev_x: S_{\bar{k}} \to B_{\bar{k}}(\tilde{G})$ is injective on $\bar{k}$-valued points. Hence it must be an isogeny and in particular the image of ev_x is a torus of the same dimension as S. This shows that rank $\Gamma \leqslant$ rank G and hence $\tilde{S} \subset \Gamma_L$ must be a maximal torus.

PROPOSITION 4.6. Suppose $B \to \mathbf{A}$ is such that for a k-rational point x_0 of $\mathbf{A}$ the restriction $B_{x_0} \to \text{Spec } k$ is trivial. Then there is a maximal torus $\tilde{T}$ of $\tilde{G}$ defined over k such that

(a) We have an embedding $\psi: \mathbf{P} \underset{k}{\times} \tilde{T} \to \overline{H}$ (where $\overline{H}$ is the extension constructed in Proposition 3.2).

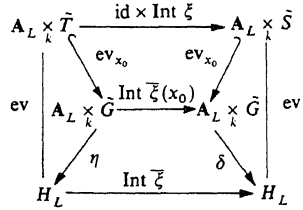
(b) When base changed to a suitable finite Galois extension L of k, $\psi_L: \mathbf{A}_L \underset{k}{\times} \tilde{T} \to H$ is the restriction of an allowable trivialisation $\mathbf{A}_L \underset{k}{\times} \tilde{G} \to H_L$.

Proof. The separably reduced Γ of the group of global sections of B is a smooth group scheme over k. Hence by Grothendieck's theorem ([2] or [8] Expose XIV) it has a maximal torus $\tilde{T}$ defined over k. By making the field L considered earlier in Proprosition 3.2 larger if necessary we can assume that $\tilde{T}_L$ and the maximal torus $\tilde{S}$ of Proposition 4.5 are conjugate by an element ξ of $\Gamma(L)$. Let $\bar{\xi}$ denote the section of $\mathbf{P}_L \to \bar{H}_L$ corresponding to ξ. We then have the commutative diagram.



Since the second vertical arrow is an embedding and the horizontal arrows are isomorphism it follows that the first vertical arrow is also an isomorphism. Since it is the pull back to L of $\mathbf{P} \underset{k}{\times} \tilde{T} \to \bar{H}$ we have proved that the latter is an embedding.

On $\mathbf{A}_L$ let $\delta: \mathbf{A}_L \underset{k}{\times} \tilde{G} \to H_L$ be the allowable trivialisation extending $ev: \mathbf{A}_L \times \tilde{S} \to H_L$ given by Proposition 3.2. Since by assumption $B_{x_0} \to \operatorname{Spec} k$ is trivial there is an allowable trivialisation $B_{x_0}(\tilde{G}) \approx \tilde{G}$. We make this identification of $B_{x_0}(\tilde{G})$ with $\tilde{G}$. We then have the commutative diagram



where η is defined to be $\operatorname{Int} \xi^{-1} \circ \delta \circ \operatorname{Int} \bar{\xi}(x_0)$. Thus η provides the allowable trivialisation extending $\mathbf{A}_L \underset{k}{\times} \tilde{T} \to H_L$. The point in this is that the torus $\tilde{T}$ which *a priori* sits in Γ gets identified as a subtorus of $\tilde{G}$ by evaluation at x_0 and the morphism of tori $\operatorname{Int} \xi: \tilde{T} \to \tilde{S}$, when $\tilde{T}$ and $\tilde{S}$ are considered as subtori of $\tilde{G}$ coincides with the restriction of the inner automorphism $\operatorname{Int} \bar{\xi}(x_0)$ of $\tilde{G}$. This completes the proof of Proposition 4.6.

We can now quickly run through the argument needed to finish the proof of theorem 1.1.

Let $B \to \mathbf{A}$ be a G-bundle which becomes trivial on $\mathbf{A}_{k_s}$, with G connected reductive. By Claim 2.9 we can assume that G is semisimple and B restricted to some k-rational point of $\mathbf{A}$ is trivial. Then Proposition 4.6 and Lemma 2.10 show that B admits a reduction of structure group to a maximal torus T of G. Since by Proposition 2.7 any T-bundle on $\mathbf{A}$ is the pull back of a T-bundle on $\operatorname{Spec} k$ this completes the proof of theorem 1.1.

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On estimates for integral solutions of linear inequalities

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Abstract. Recently, Bombieri and Vaaler obtained an interesting adelic formulation of the first and the second theorems of Minkowski in the Geometry of Numbers and derived an effective formulation of the well-known "Siegel's lemma" on the size of integral solutions of linear equations. In a similar context involving linear *inequalities*, this paper is concerned with an analogue of a theorem of Khintchine on integral solutions for inequalities arising from systems of linear forms and also with an analogue of a Kronecker-type theorem with regard to euclidean frames of integral vectors. The proof of the former theorem invokes Bombieri-Vaaler's adelic formulation of Minkowski's theorem.

Keywords. Bounds for integral solutions of linear inequalities; Theorems of Khintchine and of Kronecker-type; Bombieri-Vaaler formulation of Minkowski's theorems in geometry of numbers; Siegel's lemma

1. Introduction

Bombieri and Vaaler have recently in an interesting paper [1], obtained an adelic formulation of the first and second theorems of Minkowski in the Geometry of Numbers and as an application, through a "cube slicing inequality", derived an effective general formulation of the well-known "Siegel's lemma" on the order of magnitude of integral solutions of linear equations. In a similar context involving linear inequalities, one is led naturally to seek an analogue of a theorem of Khintchine's [4] concerning integral solutions for inequalities arising from systems of linear forms and also to look for an analogue of a theorem of Kronecker's with regard to euclidean frames of integral vectors [3]. Our proof of the above mentioned analogue of Khintchine's theorem which, incidentally, does not seem to be very accessible, according to Lekkerkerker ([7], p. 470)) invokes the adelic formulation of Minkowski's theorem, due to Bombieri and Vaaler, confirming their remark in [1]: "we certainly do not believe that our application of this inequality to Siegel's lemma exhausts its usefulness". In a preliminary version of theorem 1, we had taken into account only the archimedean primes of K; as one may see, its present formulation, however, takes care of a finite number of places of K including all the archimedean ones, as is customary with problems of this category. It is a pleasure to acknowledge here that a connected discussion some time ago with S G Dani prompted us to look for theorem 4 (in the sequel) that actually overlaps (in part) with a result of his in the section "Orbits of euclidean frames" of his interesting paper entitled "Flows in homogeneous spaces and diophantine approximation" wherein, however, different techniques are involved.

2. Terminology

If $\{F_i(x_1, \ldots, x_m, y_1, \ldots, y_r),\ i = 1, 2, \ldots, p\}$ is a given system of p real-valued functions of the real variables $x_1, \ldots, x_m, y_1, \ldots, y_r$ and if $\{\chi_i(t), i = 1, 2, \ldots, p\}$ is a

given set of p positive functions of a real variable $t > 0$, then $\{F_1, \ldots, F_p\}$ *admits the approximation* $\{\chi_1, \ldots, \chi_p\}$ if, for every real number $M > 0$, there exists $u = (u_1, \ldots, u_m) \in \boldsymbol{R}^m$ with $ht\, u := \max_i |u_i| \geqslant M$ and $v = (v_1, \ldots, v_r) \in \boldsymbol{R}^r$ with all the u_i and v_j in $\boldsymbol{Z}$ such that

$$|F_i(u_1, \ldots, u_m, v_1, \ldots, v_r)| < \chi_i(ht\, u),\ i = 1, 2, \ldots, p.$$

For example (see [6]), a homogeneous real linear form $L(x_1, \ldots, x_m, y_1) = a_1x_1 + \ldots + a_mx_m - y_1$ admits the approximation χ with $\chi(t) = 1/t^m$; the "dual" system of m linear forms $L_i(x_1, y_1, \ldots, y_m) = a_ix_1 - y_i$, $1 \leqslant i \leqslant m$, likewise admits the approximation $\{\chi, \ldots, \chi\}$ with $\chi(t) = 1/t^{1/m}$. The index ω_1' of the linear form L is defined by

$$\omega_1' = \text{l.u.b.}\ \{\omega \mid L \text{ admits the approximation } \chi \text{ with } \chi(t) = 1/t^{m+\omega}\}.$$

If, for some constant $c > 0$, the form L does *not* admit the approximation χ_1 with $\chi_1(t) = 1/(ct^{m+\omega_1'})$; then ω_1' is called the *proper index* of L. Clearly we have $0 \leqslant \omega_1' \leqslant \infty$. The form L is called *extreme*, if it has proper index 0. Thus, for an extreme form L as above, there exists a constant $c > 0$ such that, for all $u = (u_1, \ldots, u_m) \in \boldsymbol{R}^m$ with integral $u_1, \ldots, u_m$ not all 0 and for all v in $\boldsymbol{Z}$, we have

$$|a_1u_1 + \ldots + a_mu_m - v| \geqslant 1/(c\, ht\, u)^m.$$

For example, the real form $a_1x_1 - y_1$ is extreme if and only if a_1 is irrational with bounded partial quotients in its simple continued fraction expansion; in particular, for any real quadratic irrationality θ, the form $\theta x_1 - y_1$ is extreme. For the "dual" system $\{L_1, \ldots, L_m\}$ of linear forms above, the notions of the index ω_2', proper index and extreme system are defined in an entirely analogous manner (see [6]). If ω_1' (respectively ω_2') is positive, then L (respectively $\{L_1, \ldots, L_m\}$) is said to be "very well approximable", according to Schmidt [9]; the notion "extreme" corresponds to "badly approximable" in the sense of Schmidt [9]. Almost no $(a_1, \ldots, a_m)$ in $\boldsymbol{R}^m$, with regard to Lebesgue measure, has the property that the corresponding form L as above is very well approximable or badly approximable ([9], theorem 6G, p. 219). It is also known [6] that L is extreme if and only if the "dual" system $\{L_1, \ldots, L_m\}$ is extreme.

The following assertion due to Khintchine (cf. [5]) is found in a footnote to page 86 of Koksma's book [6]:

For any given m real numbers $\alpha_1, \ldots, \alpha_m$, there exists a constant $c = c(\alpha_1, \ldots, \alpha_m) > 0$ such that, for all $t \geqslant 1$ and all real numbers $\beta_1, \ldots, \beta_m$, the inequalities

$$0 < x_1 < ct^m, |\alpha_ix_1 - y_i - \beta_i| < 1/t,\ i = 1, 2, \ldots, m$$

are (simultaneously) solvable in integers $x_1, y_1, \ldots, y_m$ if and only if the system $\{L_1, \ldots, L_m\}$ with $L_i := \alpha_ix_1 - y_i (1 \leqslant i \leqslant m)$ is extreme. In §4, we consider a mild generalization of this assertion of Khintchine's and an application to estimate the magnitude of euclidean frames of integral vectors satisfying an 'irrational' system of linear inequalities.

Let K be an algebraic number field of (finite) degree d over the field $\boldsymbol{Q}$ of rational numbers and R, the ring of integers in K with a $\boldsymbol{Z}$-basis $\{\omega_1, \ldots, \omega_d\}$ to be *fixed in the sequel*. Let E be a fixed vector space of dimension l over K and let, for every place v of K, $E_v = E \otimes_K K_v$, where K_v is the completion of K at v. For archimedean v, we write $v \mid \infty$, in symbols and otherwise, we write $v \nmid \infty$. Let, for $v \nmid \infty$, R_v denote the ring of integers in K_v and M_v, a K_v-lattice in E_v (*i.e.* an open compact R_v-module contained in E_v) such

that $M_v = R_v^l$ for all but finitely many such v. For each $v|\infty$, let $\mathscr{S}_v$ be a non-empty and bounded open convex set in $E_v = K_v^l$ which is, in addition, 0-symmetric *i.e.* for every α in K_v with v-adic value $|\alpha|_v = 1$ and every x in $\mathscr{S}_v$, αx is also in $\mathscr{S}_v$. With M_v given as above for every $v \nmid \infty$, let

$$\mathscr{S} := \prod_{v|\infty} \mathscr{S}_v \times \prod_{v\nmid\infty} M_v.$$

Then $\mathscr{S}$ is an open relatively compact neighbourhood of 0 in $E_A := E \otimes_K K_A \approx K_A^l$ where K_A is the ring of K-adeles [10]. Since E is a discrete subgroup of E_A, it follows that $\mathscr{S} \cap E$ is finite. For real $\lambda > 0$, let $\lambda\mathscr{S}_v := \{\lambda x | x \in \mathscr{S}_v\}$ for $v|\infty$ and let

$$\lambda\mathscr{S} := \prod_{v|\infty} \lambda\mathscr{S}_v \times \prod_{v\nmid\infty} M_v.$$

The *successive minima* of $\mathscr{S}$ with respect to E (see [1]) are defined, for $1 \leqslant j \leqslant l$, by

$$\lambda_j := \inf \{\lambda > 0 | \lambda\mathscr{S} \cap E \text{ contains } j \text{ linearly independent vectors}\},$$

and moreover, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_l < \infty$. For $v|\infty$, let $r_v = 1$ or $1/2$ according as $K_v = \boldsymbol{R}$ or $\boldsymbol{C}$ and let $r_v^l \, dx_v$ denote Lebesgue measure in K_v^l. For $v \nmid \infty$, let dx_v denote the normalized measure in K_v^l for which R_v^l has measure $|D_v|_v^{dl/2}$ where $|D_v|_v$ is the (v-adic) value of the local "different" D_v in R_v. If Δ is the discriminant of K, then $|\Delta| = \prod_{v\nmid\infty} |D_v|_v^{-d}$. Let vol $(\mathscr{S})$ denote the volume of $\mathscr{S}$ with respect to the product measure $\prod_v dx_v$ in E_A. Then we know, from Bombieri and Vaaler [1], that

$$\lambda_1 \ldots \lambda_l \leqslant 2^l/(\text{vol}\,(\mathscr{S}))^{1/d} = \rho_E(\mathscr{S}) = \rho_E, \text{ say.}$$

For a fixed *finite* set V of (mutually inequivalent) valuations (or places) v of K containing all v with $v|\infty$, let $K_V := \prod_{v\in V} K_v$. Let us write $V_\infty = \{v | v \text{ archimedean}\}$ and $V_f = \{v \in V | v \nmid \infty\}$ so that $V = V_\infty \amalg V_f$. The ring $K(V)$ of V-integers in K is, by definition, the subring of x in K integral at all v not in V; it has the 'standard' imbedding in K_V as a discrete subring, with the quotient $F_V = K_V/K(V)$ compact [2]. As a 'fundamental set' F_V', we can take $F_\infty \times \prod_{v\in V_f} R_v$ where $F_\infty := \{a = (\ldots, a_v, \ldots) \in \prod_{v|\infty} K_v | a_v = \sum_{1\leqslant i\leqslant d} c_i \omega_i^{(v)}$ with $c_1, \ldots, c_d$ in $\boldsymbol{R}$, $\max_i |c_i| \leqslant 1/2$ and for ω_i in the fixed $\boldsymbol{Z}$-basis of R above, $\omega_i^{(v)}$ denotes the image of ω_i under the isomorphism from K to $K_v\}$. When V_f is empty, $V = V_\infty$ and we simply write K_∞ for K_V; then $K(V) = R$. For any x in K_V, the V-integer nearest to x is an element y in $K(V)$, generally unique, such that $x - y$ is in the fundamental set F_V' above; a similar definition applies, if we take a finitely generated submodule M, instead of $K(V)$, in K. For $x = (\ldots, x_v, \ldots)$ in K_V, we write $\|x\|_V$ for $\max_{v\in V} |x_v|_v$. In particular, for $V = V_\infty$, we write $\|x\|_\infty$ instead of $\|x\|_V$, for x in K_∞. For v in V_f, we fix $p_v > 0$ in $\boldsymbol{R}$ such that the corresponding "value group" is just $\{p_v^n | n \in \boldsymbol{Z}\}$. We use (small or capital) boldface letters to denote vectors or columns and corresponding letters in italics with subscripts to denote their entries: *e.g.* $\mathbf{x}$ stands for a column with entries say, $x_1, \ldots, x_m$ and $\mathbf{S}$ stands for a column with entries $S_1, \ldots, S_r$, say. With this notation for $\mathbf{x}$, we write $\|\mathbf{x}\|_V$

for $\max_i \|x_i\|_V$ when $x_1, \ldots, x_m$ are in K_V; a similar remark applies to $\|\mathbf{x}\|_v$ when $x_1, \ldots, x_r$ are all in K_v or to $\|\mathbf{S}\|_V$, $\|\mathbf{S}\|_v$ etc. The transpose of a matrix A is denoted by ${}^tA$. If $\varepsilon = (\ldots, \varepsilon_v, \ldots) \in K_V$ with rational $\varepsilon_v > 0$ for every v in V and further, if $\mathbf{a} = {}^t(a_1, \ldots, a_r)$ with all $a_i = (\ldots, a_{i,v}, \ldots)$ in K_V satisfying the condition $\|\mathbf{a}\|_v := \max_{1 \leqslant i \leqslant r} |a_{i,v}|_v \leqslant \varepsilon_v$ for every v in V, then we write

$$\mathbf{a} \underset{V}{\leqslant} \varepsilon.$$

3. An analogue of a theorem of Khintchine's

Let $S_1, \ldots, S_r$ be r linear forms in $m+r$ variables $x_1, \ldots, x_m, y_1, \ldots, y_r$ with coefficients θ_{ij} in K_V given by

$$\mathbf{S} = {}^t\Theta\,\mathbf{x} - \mathbf{y}$$

where $\mathbf{S} := {}^t(S_1 \ldots S_r)$, $\mathbf{x} = {}^t(x_1 \ldots x_m)$, $\mathbf{y} = {}^t(y_1 \ldots y_r)$ and $\Theta = (\theta_{ij})$ is an (m, r) matrix with θ_{ij} $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant r)$ as entries. Associated to $\mathbf{S}$ we have a 'dual' system of m linear forms $T_i = T_i(\mathbf{a}, \mathbf{b})$ with $\mathbf{a} = {}^t(a_1 \ldots a_r)$ and $\mathbf{b} = {}^t(b_1 \ldots b_m)$ given by

$$\mathbf{T} := {}^t(T_1 \ldots T_m) = \Theta\,\mathbf{a} + \mathbf{b}.$$

If $\mathbf{a}'$ is defined by ${}^t\mathbf{a}' = ({}^t\mathbf{a}\,{}^t\mathbf{b})$, we also write $T_i(\mathbf{a}')$ instead of $T_i(\mathbf{a}, \mathbf{b})$. When $\mathbf{a}' := {}^t(a_1' \ldots a_{r+m}')$ has all its entries in $K(V)$, we define, for given $\mathbf{u} := {}^t(u_1 \ldots u_r)$ with u_i in K_V, $\mathscr{A}(\mathbf{a}') = \mathscr{A}_{\mathbf{u}}(\mathbf{a}') := (\mathscr{A}_{\mathbf{u}}(\mathbf{a}')_v)_{v \in V} := \left(\left| \sum_{1 \leqslant j \leqslant r} u_j a_j' - c \right|_v \right)_{v \in V}$ where c is "the V-integer nearest to" $\sum_{1 \leqslant j \leqslant r} u_j a_j'$. Then clearly $\max_{v \in V} \left| \sum_j u_j a_j' - c \right|_v \leqslant \max_{v \in V} |\Sigma u_j a_j' - c'|_v$ for every V-integer c' in K. In a similar way, we can also define, for given u and for every finitely generated $K(V)$-module $M \subset K$,

$$\mathscr{A}(\mathbf{a}'; M) = \mathscr{A}_{\mathbf{u}}(\mathbf{a}'; M) := \left(\left| \sum_{1 \leqslant j \leqslant r} u_j a_j' - c^* \right|_v \right)_{v \in V}$$

where now c^* is the element of M "nearest to" $\sum_{1 \leqslant j \leqslant r} u_j a_j'$. Finally, we set, for $v \in V$,

$$\mathscr{T}(\mathbf{a}')_v = \mathscr{T}(\mathbf{a}, \mathbf{b})_v := \max_{1 \leqslant i \leqslant m} |T_i(\mathbf{a}, \mathbf{b})|_v$$

We are now in a position to state the following analogue of a result of Khintchine's [4]:

THEOREM 1. Let $S_1, \ldots, S_r$ be linear forms in $l := m+r$ variables $x_1, \ldots, x_m, y_1, \ldots, y_r$ as above, with coefficients θ_{ij} in K_V, φ a monotonic increasing function of a positive real variable t and for every v in V_f, $\tau_v \leqslant 0$ be given in $\mathbf{Z}$. Let, further, $\varepsilon = (\varepsilon_v)$ and $\delta = (\delta_v)$ in K_V be given by $\varepsilon_v = c_1/t$, $\delta_v = c_2 \varphi(t)$ for $v \mid \infty$, $\varepsilon_w = \delta_w = c_3 p_w^{\tau_w}$ (with positive constants c_1, c_2 and c_3) for $w \in V_f$. Then, for every ${}^t\mathbf{u}$ as above in K_V^r and every such ε and δ, the system of inequalities

$$\mathbf{S} - \mathbf{u} \underset{V}{\leqslant} \varepsilon$$

$$\mathbf{x} \underset{V}{\leqslant} \delta$$

admits as solutions columns $\mathbf{x}$, $\mathbf{y}$ with entries $x_1, \ldots, x_m$ and $y_1, \ldots, y_r$ respectively from a finitely generated $K(V)$-module M contained in K, provided that, for every $\mathbf{a}$ in $K(V)^r$ and $\mathbf{b}$ in $K(V)^m$ as above, the following conditions are fulfilled:

(i) $\mathscr{A}_u(\mathbf{a}, \mathbf{b}) = 0$, whenever $\|\mathbf{a}\|_v \mathscr{T}(\mathbf{a}, \mathbf{b})_v = 0$

(ii) $\mathscr{A}_u(\mathbf{a}, \mathbf{b}) \leqslant \gamma$ with $\gamma = (\gamma_v) \in K_V$ given by

$$\gamma_v := \begin{cases} c_v \|\mathbf{a}\|_\infty / \psi(\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}, \mathbf{b})_\infty) \text{ for } v \mid \infty \\ c_v p_v^{\tau_v} \max(\|\mathbf{a}\|_v, \mathscr{T}(\mathbf{a}, \mathbf{b})_v) \text{ for } v \in V_f \end{cases} \tag{2}$$

for suitable constants $c_v > 0$ and ψ defined by $\psi(t\varphi(t)) = t$.

Proof. Our proof is on the same lines as Khintchine's [4].
For v in V, let $T_{1,v}, \ldots, T_{m,v}$ denote the linear forms in the l variables in $(a_1' a_2' \ldots a_l') = {}^t\mathbf{a}' = ({}^t\mathbf{a}\ {}^t\mathbf{b})$ with the v-adic components $\theta_{ij,v}$ of θ_{ij} in K_v as coefficients in lieu of the coefficients θ_{ij} of $T_1, \ldots, T_m$. With a positive parameter μ, let us define, for every v in V, linear forms $L_{1,v}, \ldots, L_{l,v}$ in $a_1' = a_1, \ldots, a_r' = a_r, a_{r+1}' = b_1, \ldots, a_l' = b_m$ by

$$L_{i,v} = \mu_v^r T_{i,v},\ L_{m+j,v} = \mu_v^{-m} a_j \ (1 \leqslant i \leqslant m;\ 1 \leqslant j \leqslant r) \tag{3}$$

where $\mu_v := \mu$ for $v \mid \infty$ and $\mu_v = 1$, otherwise. The determinant of this system of l linear forms is of absolute value 1, for every v in V. Let $\mathscr{S}_v := \{(x_1, \ldots, x_l) \in K_v^l \mid L_{i,v}(x_1, \ldots, x_l)|_v \leqslant 1 \text{ for } 1 \leqslant i \leqslant l\}$ for every v in V and $\mathscr{S} := \prod_{v \in V} \mathscr{S}_v \times \prod_{v \notin V} R_v^l$. If $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_l < \infty$ are the l successive minima of $\mathscr{S}$, there exists a corresponding set $\{\mathbf{a}'(1), \ldots, \mathbf{a}'(l)\}$ of linearly independent vectors in $K(V)^l$ such that $\mathbf{a}'(j)$ is in $\lambda\mathscr{S} \cap K^l$ for all $\lambda > \lambda_j$ and further $\lambda_j = \max_{1 \leqslant i \leqslant l} |L_{i,v}(\mathbf{a}'(j))|_v$ for $v \mid \infty$. From [1], we know that

$$\lambda_1 \ldots \lambda_l \leqslant 2^l / \mathrm{vol}(\mathscr{S})^{1/d} = \rho_E(\mathscr{S}) = \rho_E, \text{ say.}$$

In the sequel, we shall write ${}^t\mathbf{a}'(j) = ({}^t\mathbf{a}(j)\ {}^t\mathbf{b}(j))$ with ${}^t\mathbf{a}(j) \in K^r$ and ${}^t\mathbf{b}(j) \in K^m$ for $1 \leqslant j \leqslant l$. From (3) and the construction of $\mathbf{a}'(j)$, it is immediate that

$$\|\mathbf{a}(j)\|_v \leqslant \begin{cases} \mu^m \lambda_j & \text{for } v \mid \infty \\ 1 & \text{for } v \in V_f \end{cases} \tag{4}$$

and

$$\mathscr{T}(\mathbf{a}'(j))_v \leqslant \begin{cases} \mu^{-r} \lambda_j & \text{for } v \mid \infty \\ 1 & \text{for } v \in V_f. \end{cases} \tag{5}$$

We now claim that, as a consequence of conditions (i)–(ii) in (2), we have, for $1 \leqslant j \leqslant l$

$$\mathscr{A}_u(\mathbf{a}'(j)) \underset{V}{\leqslant} \gamma' \text{ with } \gamma' = (\gamma_v') \text{ and } \gamma_v' = \begin{cases} c_v \lambda_j \mu^m / \psi(\mu^l) & \text{for } v \mid \infty \\ c_v p_v^{\tau_v} & \text{for } v \in V_f. \end{cases} \tag{6}$$

In view of (4) and (5), only the part of the claim pertaining to $v \mid \infty$ in (6) needs to be established. If $\|\mathbf{a}(j)\|_\infty \mathscr{T}(\mathbf{a}'(j))_\infty = 0$, then there is nothing to prove. Let then condition (ii) in (2) hold, with $\|\mathbf{a}(j)\|_\infty \mathscr{T}(\mathbf{a}'(j))_\infty > 0$. If $\sigma_j := \|\mathbf{a}(j)\|_\infty / \mathscr{T}(\mathbf{a}'(j)) \geqslant \mu^l$, then the required inequality follows at once from (4) and condition (ii), in view of the monotonicity of ψ. On the other hand, if $\sigma_j < \mu^l$, then $\sigma_j / \psi(\sigma_j) = \varphi(\psi(\sigma_j))$

$\leqslant \varphi(\psi(\mu^l)) = \mu^l/\psi(\mu^l)$. Again, it follows from (ii) and (5) that for $v|\infty$,

$$\gamma_v \leqslant c_v \mathscr{T}(\mathbf{a}'(j))_\infty \mu^l/\psi(\mu^l) \leqslant \gamma'_v$$

proving our claim.

Since $\mathbf{a}'(1), \ldots, \mathbf{a}'(l)$ are linearly independent,

$$\mathscr{B} := \det(\mathbf{a}'(1) \ldots \mathbf{a}'(l)) \in K \setminus \{0\}.$$

Assuming all valuations v to be suitably normalized, as in [1], we have $\prod_v |\mathscr{B}|_v = 1$, by the product formula. The linear forms $L_{i,v}$ (for $1 \leqslant i \leqslant l$) have determinant 1 in absolute value, for every v in V and therefore

$$|\mathscr{B}|_v = |\det(L_{i,v}(\mathbf{a}'(j))_{1 \leqslant i,j \leqslant l})|_v$$

In the light of the definition of the vectors for successive minima, this gives immediately the inequalities

$$|\mathscr{B}|_v \leqslant \begin{cases} l!\lambda_1 \ldots \lambda_l \leqslant l! 2^l/\mathrm{vol}(\mathscr{S})^{1/d} = l!\rho_E \text{ for } v|\infty \\ 1 \quad \text{for} \quad v \nmid \infty. \end{cases} \tag{7}$$

For v-adic measures normalized as in [1], we note that $\mathrm{vol}(\mathscr{S}) = 2^{ld}(\pi/2)^{ls}|\Delta|^{-l/2}$ where s is the number of complex places of K. From the product formula and (7), we derive at once the following estimates:

$$1 = \prod_v |\mathscr{B}|_v \leqslant |\mathscr{B}|_{v_1} \times \begin{cases} (l!\rho_E)^{d-1} & \text{for} \quad v_1|\infty \\ (l!\rho_E)^d & \text{for} \quad v_1 \nmid \infty \end{cases}$$

i.e.

$$|\mathscr{B}|_v \geqslant \theta_v := \begin{cases} (l!\rho_E)^{1-d} & \text{for} \quad v|\infty \\ (l!\rho_E)^{-d} & \text{for} \quad v \nmid \infty. \end{cases} \tag{8}$$

By (7), $\mathscr{B}$ is in $R \setminus \{0\}$ and there exists a possibly non-empty *finite* set W of valuations v of K with $v \nmid \infty$, depending only on K, l and V, such that $|\mathscr{B}|_v = 1$ for all $v \nmid \infty$ with $v \notin W$.

Let us next consider the expansion

$$\det(L_{j,v}(\mathbf{a}'(i))) = \sum_\sigma \pm L_{1,v}(\mathbf{a}'(l_1)) \ldots L_{m,v}(\mathbf{a}'(l_m)) D(\sigma)_v$$

where the summation is over all the distinct subsets $\sigma = \{l_1, \ldots, l_m\}$ consisting of m (distinct) elements in $\{1, \ldots, l\}$ and for any such σ, $D(\sigma)_v$ is the determinant of the r-rowed square matrix obtained by deleting the rows with indices $i = l_1, \ldots, l_m$ from the matrix constituted by the last r columns of $(L_{j,v}(\mathbf{a}'(i))$ corresponding to the column indices $j = m+1, m+2, \ldots, m+r(=l)$. The number of such subsets σ being $l!/(r!)$, the (v-adic) value of the term corresponding to *at least one* of these subsets is $\geqslant r!\theta_v/(l!)$ for $v|\infty$ and $\geqslant \theta_v$ for $v \in V_f$, in view of (8). Let us fix one of these subsets, say, $\sigma_0 = \{q_1, \ldots, q_m\}$ which, of course, depends on v and let $\{1, \ldots, l\} \setminus \sigma_0 = \{s_1, \ldots, s_r\}$. Thus, if

$$\mathscr{M}_v := |\det(L_{m+j,v}(\mathbf{a}'(i)))_{\substack{1 \leqslant j \leqslant r \\ i \notin \sigma_0}}|_v,$$

then

$$\mathscr{M}_v \prod_{1 \leqslant i \leqslant m} |L_{i,v}(\mathbf{a}'(q_i))|_v \geqslant \begin{cases} r!\theta_v/(l!) & \text{for} \quad v|\infty \\ \theta_v & \text{for} \quad v \in V_f. \end{cases}$$

From the nature of the construction of $\mathbf{a}'(q_i)$, this gives us

$$\mathscr{M}_v \geqslant \begin{cases} r!\theta_v/(l!\lambda_{q_1}\ldots\lambda_{q_m}) & \text{for } v|\infty \\ \theta_v & \text{for } v\in V_f. \end{cases}$$

Writing

$$\mathbf{a}'(i) = {}^t(\alpha_{i,1}\ldots\alpha_{i,l})\,(1\leqslant i\leqslant l),\ \mathscr{D}_v := |\det(\alpha_{ij})_{\substack{1\leqslant j\leqslant r\\ i\notin\sigma_0}}|_v,$$

we see from (3) that $\mathscr{D}_v = \mu_v^{mr}\mathscr{M}_v$ for every v in V. Therefore, we have the lower bounds

$$\mathscr{D}_v \geqslant \begin{cases} r!\theta_v\mu^{mr}/(l!\lambda_{q_1}\ldots\lambda_{q_m}) & \text{for } v|\infty \\ \theta_v & v\in V_f. \end{cases} \tag{9}$$

In order to obtain the required bounds for $\mathbf{S}-\mathbf{u}$ and $\mathbf{x}$ under the assumption of conditions (i)–(ii) in (2), let us consider

$$\begin{aligned}\eta_i &:= {}^t\mathbf{a}'(i)\,(\mathbf{S}-\mathbf{u})\,(1\leqslant i\leqslant l)\\ &= \sum_{1\leqslant j\leqslant r}\alpha_{i,j}\Big(\sum_{1\leqslant k\leqslant m}(\theta_{kj}x_k - y_j - u_j)\Big)\\ &= \sum_{1\leqslant k\leqslant m}x_kT_k(\mathbf{a}'(i)) - \sum_{1\leqslant k\leqslant m}\alpha_{i,r+k}x_k - \sum_{1\leqslant j\leqslant r}\alpha_{i,j}y_j\\ &\qquad - \sum_{1\leqslant j\leqslant r}\alpha_{i,j}u_j.\end{aligned}$$

Observe now that it is possible to find $x_1,\ldots,x_m,y_1,\ldots,y_r$ in K (uniquely, in view of $\mathscr{B}\neq 0$) such that

$$\sum_{1\leqslant k\leqslant m}\alpha_{i,r+k}x_k + \sum_{1\leqslant j\leqslant r}\alpha_{i,j}y_j = v_i\,(1\leqslant i\leqslant l) \tag{10}$$

where, for $1\leqslant i\leqslant l$, v_i is the V-integer nearest to $-\sum_{1\leqslant j\leqslant r}\alpha_{i,j}u_j$. The elements $x_1,\ldots,x_m,y_1,\ldots,y_r$ constituting the solution belong to the fixed $K(V)$-module $\mathscr{B}^{-1}K(V)$ (independent of $\mathbf{u}$) and are integral at all the non-archimedean v outside the finite set W (described after the derivation of (8)). Then, for $1\leqslant i\leqslant l$, we have

$$\sum_{1\leqslant j\leqslant r}\alpha_{i,j}(S_j-u_j) = \eta_i = \sum_{1\leqslant k\leqslant m}x_kT_k(\mathbf{a}'(i)) - \sum_{1\leqslant j\leqslant r}\alpha_{i,j}u_j - v_i. \tag{11}$$

Solving for $x_1,\ldots,x_m$ from (10) and working in the field K_v for a fixed v in V, we have

$$x_j\cdot\mathscr{B} = \det{}^t(\boldsymbol{\beta}_1\ldots\boldsymbol{\beta}_l) = \det(\boldsymbol{\beta}_1\ldots\boldsymbol{\beta}_l) = \det(\boldsymbol{\beta}_1'\ldots\boldsymbol{\beta}_l') = \mathscr{C},\ \text{say},$$

where the columns $\boldsymbol{\beta}_i\cdot\boldsymbol{\beta}_i'$ are respectively given by

$${}^t\boldsymbol{\beta}_i = (\alpha_{i,1}\ldots\alpha_{i,r}\alpha_{i,r+1}\ldots\alpha_{i,r+j-1}\ v_i\,\alpha_{i,r+j+1}\ldots\alpha_{i,r+m}),$$

$${}^t\boldsymbol{\beta}_i' = (\alpha_{i,1}\ldots\alpha_{i,r}T_1(\mathbf{a}'(i))\ldots T_{j-1}(\mathbf{a}'(i))$$

$$v_i + \sum_k\alpha_{i,k}u_kT_{j+1}(\mathbf{a}'(i))\ldots T_m(\mathbf{a}'(i))),$$

taking the liberty of omitting the subscript v from $T_{j,v}\,(1 \leqslant j \leqslant m)$ and $u_{k,v}\,(1 \leqslant k \leqslant r)$. The general term in the expansion of det $(\boldsymbol{\beta}'_1 \ldots \boldsymbol{\beta}'_l)$ is of the form

$$\pm \alpha_{t_1,1} \ldots \alpha_{t_r,r} \left(\prod_{1 \leqslant n \leqslant m} T_n(\mathbf{a}'(\mathbf{t}_{r+n})) \right) / T_j(\mathbf{a}'(t_{r+j}))\,(v_{t_{r+j}} + \sum_k \alpha_{t_{r+j},k} u_k)$$

where $(t_1, \ldots, t_l)$ is a permutation of $(1, 2, \ldots, l)$. Because of our special choice of v above, we have

$$\left(\left| v_i + \sum_{1 \leqslant k \leqslant r} \alpha_{i,k} u_k \right|_v \right)_{v \in V} = \mathscr{A}(\mathbf{a}'(i)) \quad \text{for} \quad 1 \leqslant i \leqslant l.$$

Using now the inequalities in (4)–(6), we obtain

$$|\mathscr{C}|_v \leqslant \begin{cases} l!\lambda_1 \ldots \lambda_l \mu^r c_v \mu^m / \psi(\mu^l) & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} & \text{for} \quad v \in V_f \end{cases}$$

and in view of (8), we can deduce that, for $1 \leqslant k \leqslant m$,

$$|x_k|_v \leqslant \begin{cases} l!\lambda_1 \ldots \lambda_l c_v \mu^l / (\theta_v \psi(\mu^l)) & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} / \theta_v & \text{for} \quad v \in V_f. \end{cases} \tag{12}$$

From (11), (12), (5) and (6), it then follows easily for $1 \leqslant i \leqslant l$ that

$$\begin{aligned} |\eta_i|_v &\leqslant \begin{cases} \sum_{1 \leqslant k \leqslant m} |x_k|_v \mathscr{T}(\mathbf{a}'(i))_v + \mathscr{A}(\mathbf{a}'(i))_v & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} \max(1, 1/\theta_v) & \text{for} \quad v \in V_f \end{cases} \\ &\leqslant \begin{cases} l!m\lambda_1 \ldots \lambda_l c_v (\mu^l / (\theta_v \psi(\mu^l))) (\lambda_i / \mu^r) + c_v \lambda_i \mu^m / \psi(\mu^l) & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} \max(1, 1/\theta_v) & \text{for} \quad v \in V_f \end{cases} \\ &\leqslant \begin{cases} c_v (l!m\rho_E/\theta_v + 1) \mu^m \lambda_i / \psi(\mu^l) & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} \max(1, 1/\theta_v) & \text{for} \quad v \in V_f. \end{cases} \end{aligned} \tag{13}$$

Reading off the values of $S_j - u_j\,(1 \leqslant j \leqslant r)$ from the equations on the left half of (11) with $i = s_1, \ldots, s_r, \ldots$, we obtain the estimates

$$\mathscr{D}_v |S_j - u_j|_v \leqslant \begin{cases} (r-1)! \sum_{1 \leqslant i \leqslant r} |\eta_{s_i}|_v \left(\prod_{1 \leqslant k \leqslant r} \|\mathbf{a}(s_k)\|_v \right) \Big/ \|\mathbf{a}(s_i)\|_v & \text{for} \quad v|\infty \\ \max_{1 \leqslant i \leqslant r} |\eta_{s_i}|_v \prod_{1 \leqslant k \leqslant r} \|\mathbf{a}(s_k)\|_v / \|\mathbf{a}(s_i)\|_v & \text{for} \quad v \in V_f. \end{cases}$$

Applying (13), (4) and (9) next, we have

$$|S_j - u_j|_v \leqslant \begin{cases} c_v \rho_E (l!/\theta_v)(1 + m\rho_E(l!)/\theta_v)/\psi(\mu^l) & \text{for} \quad v|\infty \\ c_v p_v^{\tau_v} \max(1, 1/\theta_v) & \text{for} \quad v \in V_f \end{cases}$$

and

$$|x_k|_v \leqslant \begin{cases} c_v \rho_E (l!/\theta_v) \mu^l / \psi(\mu^l) & \text{for} \quad v|\infty \\ (c_v/\theta_v) p_v^{\tau_v} & \text{for} \quad v \in V_f \end{cases}$$

for $1 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant m$. Let us now take

$$c_1 = c_v \rho_E (l!/\theta_v)(1 + m\rho_E(l!)/\theta_v),\ c_2 = c_v \rho_E (l!)/\theta_v,\ c_3 = c_v \max(1, 1/\theta_v).$$

We may note further that for any given $t > 0$, there exists $\mu > 0$ with $\psi(\mu^l) = t$. Then th

existence of $x_1, \ldots, x_m, y_1, \ldots, y_r$ in the module $\mathscr{B}^{-1} K(V)$ satisfying the given system of inequalities for $\mathbf{S} - \mathbf{u}$ and $\mathbf{x}$ is established as a consequence of conditions (i)–(ii) in (2).

Towards a converse (to theorem 1), we have the following

THEOREM 2. With notation as in theorem 1, let for $\mathbf{S}$, $\mathbf{u}$ and every ε and δ, the inequalities $\mathbf{S} - \mathbf{u} \underset{V}{\leqslant} \varepsilon$, $\mathbf{x} \underset{V}{\leqslant} \delta$ be solvable with columns $\mathbf{x}$, $\mathbf{y}$ having entries in a finitely generated $K(V)$-module $M \subset K$. Then, for every ${}^t\mathbf{a}' = ({}^t\mathbf{a}\ {}^t\mathbf{b})$ with ${}^t\mathbf{a}$ in $K(V)^r$ and ${}^t\mathbf{b}$ in $K(V)^m$, we have

(i) $\mathscr{A}_{\mathbf{u}}(\mathbf{a}'; M) = 0$, whenever $\|\mathbf{a}\|_V \mathscr{T}(\mathbf{a}, \mathbf{b})_V = 0$ and

(ii) $\|\mathscr{A}_{\mathbf{u}}(\mathbf{a}'; M)\|_V \leqslant \|\gamma\|_V$ for $\gamma = (\gamma_v) \in K_V$ where

$$\gamma_v := \begin{cases} c_v \|\mathbf{a}\|_\infty / \psi(\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}, \mathbf{b})_\infty) & \text{for } v | \infty \\ c_v p_v^{\tau_v} \max(\|\mathbf{a}\|_v, \mathscr{T}(\mathbf{a}, \mathbf{b})_v) & \text{for } v \in V_f \end{cases}$$

with positive constants c_v for every v in V.

Proof. For a given $\mathbf{x} = {}^t(x_1 \ldots x_m)$, $\mathbf{y} = {}^t(y_1 \ldots y_r)$ satisfying the inequalities $\mathbf{S} - \mathbf{u} \leqslant \varepsilon$, $\mathbf{x} \leqslant \delta$ corresponding to the given ε, δ and $\mathbf{u}$ and for the given $\mathbf{a} = {}^t(a_1 \ldots a_r)$, $\mathbf{b} = {}^t(b_1 \ldots b_m)$, let us define $v = \sum_{1 \leqslant j \leqslant r} a_j y_j + \sum_{1 \leqslant i \leqslant m} b_i x_i$ which is clearly in M. Then, as in (11), we have

$$\sum_{1 \leqslant j \leqslant r} a_j (S_j - u_j) = \sum_{1 \leqslant k \leqslant m} x_k T_k(\mathbf{a}') - \sum_{1 \leqslant j \leqslant r} a_j u_j - v$$

and

$$\begin{aligned} \mathscr{A}(\mathbf{a}'; M)_V &\leqslant \left\| \sum_{1 \leqslant j \leqslant r} a_j u_j + v \right\|_V \\ &= \left\| \sum_{1 \leqslant j \leqslant r} a_j (S_j - u_j) - \sum_{1 \leqslant k \leqslant m} x_k T_k(\mathbf{a}') \right\|_V \\ &\leqslant \max_{\substack{v | \infty \\ w \in V_f}} \left\{ \begin{matrix} r \|\mathbf{a}\|_v c_1 / t + m c_2 \varphi(t) \mathscr{T}(\mathbf{a}')_v \\ \|\mathbf{a}\|_w c_3 p_w^{\tau_w}, c_3 p_w^{\tau_w} \mathscr{T}(\mathbf{a}')_w \end{matrix} \right\} \end{aligned}$$

If $\|\mathbf{a}\|_V = 0$, then $\mathbf{a} = 0$ and so $\mathscr{A}(\mathbf{a}'; M)_V = 0$. On the other hand, if $\mathscr{T}(\mathbf{a}')_V = 0$, then the validity of

$$\mathscr{A}(\mathbf{a}'; M)_v \leqslant \max(r \|\mathbf{a}\|_v c_1 / t, \|\mathbf{a}\|_w c_3 p_w^{\tau_w})$$

for every $t > 0$ and every $\tau_w \leqslant 0$ in $\boldsymbol{Z}$ forces $\mathscr{A}(\mathbf{a}'; M)_V$ to be 0. If therefore $\|\mathbf{a}\|_\infty \mathscr{T}(\mathbf{a}')_\infty > 0$, we can find $t > 0$ such that $t\varphi(t) = \|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}')_\infty$ i.e. $t = \psi(\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}')_\infty)$ and then $\mathscr{T}(\mathbf{a}')_\infty \varphi(t) = \|\mathbf{a}\|_\infty / \psi(\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}')_\infty)$. Thus, in any case,

$$\mathscr{A}(\mathbf{a}'; M)_\infty \leqslant (r c_1 + m c_2) \|\mathbf{a}\|_\infty / \psi(\|\mathbf{a}\|_\infty / \mathscr{T}(\mathbf{a}')_\infty)$$

implying (i)–(ii) for $v | \infty$. Consequently, for every v in V,

$$\mathscr{A}(\mathbf{a}'; M)_v \leqslant \mathscr{A}(\mathbf{a}'; M)_V \leqslant \max_{v \in V} |\gamma_v|$$

with $c_v = r c_1 + m c_2$ for $v | \infty$ and $c_w = c_3$ for $w \in V_f$.

4. Applications

We apply the foregoing to obtain a mild generalization of an assertion of Khintchine ([6], page 86) referred to earlier and a quantitative version of Kronecker's theorem concerning bounds for the size of frames of integral vectors in euclidean space which arise as solutions of linear inequalities given by an 'irrational' system of linear forms.

Let $\vartheta_1, \ldots, \vartheta_r$ in K_A be linearly independent over K. The homogeneous form $L = \vartheta_1 x_1 + \ldots + \vartheta_m x_m - y$ has K-index $\geqslant 0$ in a sense quite analogous to that of [6], in view of [2]. Let us now suppose that L has *finite* proper K-index $v \geqslant 0$, in the following sense: namely, that for any given $\varepsilon > 0$ and for any $t > 0$, there exist $u_1', \ldots, u_r', v'$ not all 0 in R such that

$$\| \vartheta_1 u_1' + \ldots + \vartheta_r u_r' - v' \|_\infty < 1/t^{r+v-\varepsilon}, \quad \max_{1 \leqslant i \leqslant r} \| u_i' \|_\infty \leqslant t$$

but there exists $c = c(\vartheta_1, \ldots, \vartheta_r, K)$ such that

$$\| \vartheta_1 a_1 + \ldots + \vartheta_r a_r - b \|_\infty \geqslant 1/\left(c \max_{1 \leqslant i \leqslant r} \| a_i \|_\infty \right)^{r+v} \tag{14}$$

for all $a_1, \ldots, a_r, b$ (not simultaneously 0) in R. We are then in a position to appeal to theorem 1, taking

$$S_j = \vartheta_j x_1 - y_j \, (1 \leqslant j \leqslant r), \; m = 1, \; V = V_\infty \text{ and } \varphi(t) = t^{r+v}.$$

It is clear that $\mathscr{A}(\mathbf{a}, b)_\infty = 0$ for $\mathbf{a} = {}^t(a_1 \ldots a_r)$, whenever $\| \mathbf{a} \|_\infty \cdot \mathscr{T}(\mathbf{a}, b)_\infty = 0$, in view of the linear independence of $\vartheta_1, \ldots, \vartheta_r$ over K. Since $\mathscr{A}(\mathbf{a}, b)_v \leqslant 1/2 \sum_{1 \leqslant i \leqslant d} \| \omega_i \|_\infty$, the first inequality in (ii) for $v | \infty$ can certainly be satisfied with $c_v = 1/2 \sum_{1 \leqslant i \leqslant d} \| \omega_i \|_\infty \, c^{(r+v)/(r+v+1)}$; indeed, (14) ensures that $\mathscr{T}(\mathbf{a}, b)_\infty \geqslant 1/(c \| \mathbf{a} \|_\infty)^{r+v}$ for $\| \mathbf{a} \|_\infty \geqslant 1$ and therefore

$$\begin{aligned} 1/2 \sum_{1 \leqslant i \leqslant d} \| \omega_i \|_\infty &\leqslant c_v \| \mathbf{a} \|_\infty / (\| \mathbf{a} \|_\infty / \mathscr{T}(\mathbf{a}, b))_\infty^{1/(r+1+v)} \\ &= c_v \| \mathbf{a} \|_\infty^{(r+v)/(r+1+v)} \mathscr{T}(\mathbf{a}, b)_\infty^{1/(r+1+v)}. \end{aligned}$$

Thus, for *every* given $u_1, \ldots, u_r$ in K_∞, the given system of inequalities for $\mathbf{S} - \mathbf{u}$ and $\mathbf{x}$ is satisfied by $x_1, y_1, \ldots, y_r$ in $\mathscr{B}^{-1} R$. Taking $\mathscr{B}^{-1} u_1, \ldots, \mathscr{B}^{-1} u_r$ in place of $u_1, \ldots, u_r$ and using constants $c_1' = c_1 \| \mathscr{B} \|_\infty$, $c_2' = c_2 \| \mathscr{B} \|_\infty$, we have

THEOREM 3. Let $\vartheta_1, \ldots, \vartheta_r$ in K_A be linearly independent over K and let the linear form $\vartheta_1 x_1 + \ldots + \vartheta_r x_r - y$ in $x_1, \ldots, x_r, y$ have (finite) proper K-index $v \geqslant 0$ in the sense described above. Then, for any $t \geqslant 1$ and any $u_1, \ldots, u_r$ in K_A, we have positive constants c_1', c_2' depending on $\vartheta_1, \ldots, \vartheta_r$ and K and further $u_1', v_1', \ldots, v_r'$ not all 0 in R such that

$$\| \vartheta_i u_1' - v_i' - u_i \|_\infty \leqslant c_1'/t, \; 1 \leqslant i \leqslant r \text{ and}$$

$$\| u_1' \|_\infty \leqslant c_2' t^{r+v}.$$

Remark. For $K = \mathbf{Q}$ and $v = 0$, this is just Khintchine's assertion ([6], p. 86) for extreme systems.

Suppose now that $B = (b_{ij})$ is an $(l-1, l)$ matrix with elements in K_∞ such that for every non-zero row ${}^t\mathbf{a} = (a_1, \ldots, a_l)$ with all a_i in K, the (l, l) matrix $A := \begin{pmatrix} B \\ {}^t\mathbf{a} \end{pmatrix}$ is invertible. Then the same property holds as well for BV, where V is an arbitrary permutation matrix. We may therefore assume, without loss of generality, that the first $l-1$ columns of B form an invertible matrix, say B_1. Writing $B = B_1(E\,\vartheta)$ with E equal to the $(l-1)$-rowed identity matrix and ${}^t\vartheta = (\vartheta_1, \ldots, \vartheta_{l-1})$, we say that $\vartheta_1, \ldots, \vartheta_{l-1}$ are "associated to" B. For every row ${}^t\mathbf{a} \neq 0$ over K, $\begin{pmatrix} B_1^{-1} & {}^t\mathbf{O} \\ \mathbf{O} & 1 \end{pmatrix} A$ is invertible, whenever A is invertible and therefore the above-mentioned property of B is equivalent to the condition that

$$1, \vartheta_1, \ldots, \vartheta_{l-1} \text{ are linearly independent over } K, \tag{15}$$

for $\vartheta_1, \ldots, \vartheta_{l-1}$ associated to B. Indeed, ${}^t\mathbf{a}$ is independent of the $l-1$ rows of $(E\vartheta)$ if and only if the relations

$$a_1 + \lambda_1' = a_2 + \lambda_2' = \ldots = a_{l-1} + \lambda_{l-1}'$$
$$= a_l + \vartheta_1\lambda_1' + \vartheta_2\lambda_2' + \ldots + \vartheta_{l-1}\lambda_{l-1}' = 0$$

hold for *no* $\lambda_1', \ldots, \lambda_{l-1}'$ in K_∞; this condition is evidently equivalent to the condition in (15). For $K = \boldsymbol{Q}$, the above-mentioned property of B is the same as the corresponding $l-1$ linear forms having rationality rank $l-1$ in the sense of Kronecker [8]. If, in this case, B has all its entries in $\boldsymbol{R} \cap \overline{\boldsymbol{Q}}$ then $\vartheta_1, \ldots, \vartheta_{l-1}$ are real algebraic numbers satisfying condition (15). For general K, we say that the $l-1$ linear forms arising from B have K-rationality rank $l-1$, whenever the associated $\vartheta_1, \ldots, \vartheta_{l-1}$ satisfy condition (15).

Corollary to theorem 3. Let $F_j = \sum_{1 \leqslant p \leqslant l} b_{j,p}x_p$, $1 \leqslant j \leqslant l-1$, be $l-1$ linear forms in variables $x_1, \ldots, x_l$ and coefficients $b_{j,p}$ from K_∞, having K-rationality rank equal to $l-1$ and let, further, the linear form $\vartheta_1 x_1 + \ldots + \vartheta_{l-1}x_{l-1} - y$ for $\vartheta_1, \ldots, \vartheta_{l-1}$ associated to $(b_{j,p})$ have (finite) proper index $\nu \geqslant 0$. Then for any $u_1^*, \ldots, u_{l-1}^*$ in K_∞ and $t > 0$, there exists constants $c_1'', c_2'' > 0$ depending, in general, only on $F_1, \ldots, F_{l-1}$ and K and integers $u_1', \ldots, u_l'$ not all 0 in K such that

$$\left\| \sum_{1 \leqslant p \leqslant l} b_{j,p}u_p' - u_j^* \right\|_\infty \leqslant c_1''/t,\ 1 \leqslant j \leqslant l-1 \text{ and}$$

$$\|u_i'\|_\infty \leqslant c_2'' t^{l-1+\nu},\ 1 \leqslant i \leqslant l.$$

Proof. After an appropriate permutation of the variables $x_1, \ldots, x_l$, if necessary and taking $l-1$ suitable linear combinations of $F_1, \ldots, F_{l-1}$, we may reduce ourselves to a system of linear forms of the type $\vartheta_1 x_1 - x_2, \ldots, \vartheta_{l-1}x_1 - x_l$ and suitable corresponding values for $u_1^*, \ldots, u_{l-1}^*$. This process being evidently reversible, the corollary is immediate from theorem 3.

Remark. The constant c_1'' in the corollary may, without loss of generality, be taken as 1, after suitably modifying t and c_2''.

As another application, we derive now a quantitative version of a classical result of

Kronecker type (Satz 62 [8], p. 148) on the existence of independent lattice points in $\boldsymbol{R}^l$ (or what is the same, of integral frames) satisfying linear inequalities arising from an "irrational" system of linear forms (*i.e.* of maximal rationality rank in the sense of Kronecker [8]).

THEOREM 4. Let $F_j = F_j(\mathbf{x}) = \sum_{1 \leqslant p \leqslant l} a_{j,p} x_p$, $1 \leqslant j \leqslant l-1$, be $l-1$ linear forms in l variables $x_1, \ldots, x_l$ with coefficients $a_{j,p}$ in K_∞, K-rationality rank $l-1$ and (finite) proper K-index $v \geqslant 0$ as described above. With $\tau_{j,m} = \tau_{j,m}(\mathbf{u}_m) := \sum_{1 \leqslant p \leqslant l} a_{j,p} u_{p,m}$ for $1 \leqslant j, m \leqslant l-1$, we can find, for any $T > 0$, independent vectors ${}^t\mathbf{u}_i = (u_{1,i} \ldots u_{l,i}) \in R^l$, $1 \leqslant i \leqslant l-1$ such that

$$\|\tau_{j,m}\|_\infty < 1/T^{1/(l-1)},\ 1 \leqslant j, m \leqslant l-1, \text{ and}$$

$$\|\mathbf{u}_q\|_\infty := \max_{1 \leqslant p \leqslant l} \|u_{p,1}\| < c_3' T^{l+v-1/(l-1)},\ 1 \leqslant q \leqslant l-1$$

for a constant $c_3' > 0$ depending only on $F_1, \ldots, F_{l-1}$ and K, in general and further such that the $(l-1, l-1)$ matrix $(\tau_{j,m})$ is invertible.

Proof. Applying the corollary to theorem 3 (with $c_1'' = 1$, as mentioned in the subsequent remark, $(u_1^*)_v = \ldots = (u_{l-1}^*)_v = 2/t$ for $v \mid \infty$ and $t = 4T^{1/(l-1)}$), there exists ${}^t\mathbf{u}_1 = (u_{1,1} \ldots u_{l,1}) \in R^l$ such that

$$1/(4T^{1/(l-1)}) \leqslant \left\| \sum_{1 \leqslant p \leqslant l} a_{j,p} u_{p,1} \right\|_\infty < 3/(4T^{1/(l-1)}),\ 1 \leqslant j \leqslant l-1$$

$$\|u_1\|_\infty \leqslant c_2^* T^{1+v/(l-1)} \tag{16}$$

for some constant c_2^*. Clearly then, every $\tau_{j,1} = \sum_{1 \leqslant p \leqslant l} a_{j,p} u_{p,1}$ is invertible in K_∞. Let us suppose, without loss of generality, that $\|\tau_{l-1,1}\|_\infty \geqslant \|\tau_{j,1}\|_\infty$ for $1 \leqslant j \leqslant l-1$. The independence of $F_1, \ldots, F_{l-1}$ implies that of $G_1, \ldots, G_{l-2}$ where, for $1 \leqslant i \leqslant l-2$,

$$G_i := F_i - (\tau_{i,1}/\tau_{l-1,1}) F_{l-1} = \sum_{1 \leqslant p \leqslant l} (a_{i,p} - (\tau_{i,1}/\tau_{l-1,1}) a_{l-1,p}) x_p.$$

Moreover, $G_1, \ldots, G_{l-2}$ have K-rationality rank $l-2$; otherwise, any non-zero linear form with coefficients in K belonging to the K_∞-linear span of $G_1, \ldots, G_{l-2}$ will also lie in the K_∞-linear span of $F_1, \ldots, F_{l-1}$, giving a contradiction. By induction on the rationality rank, there exist already $l-2$ independent vectors ${}^t\mathbf{v}_j = (v_{1,j}, \ldots, v_{l,j})$ in R^l, $1 \leqslant j \leqslant l-2$ such that, with

$$\tau_{i,j}' := \sum_{1 \leqslant p \leqslant l} (a_{i,p} - (\tau_{i,1}/\tau_{l-1,1}) a_{l-1,p}) v_{p,j},\ 1 \leqslant i \leqslant l-2,$$

we have

$$\|\tau_{i,j}'\|_\infty < 1/(4T^{1/(l-1)})\ 1 \leqslant i, j \leqslant l-2,$$

$$\|v_{p,j}\|_\infty \leqslant c_3'' T^{v_{l-2}} \quad (1 \leqslant j \leqslant l-2;\ 1 \leqslant p \leqslant l), \tag{17}$$

$$\det(\tau_{i,j}') \neq 0$$

for an exponent v_{l-2} and a suitable constant $c_3'' > 0$. Evidently

$$\sum_{1 \leqslant p \leqslant l} (a_{i,p} - (\tau_{i,1}|\tau_{l-1,1})a_{l-1,p})u_{p,1} = 0,\ 1 \leqslant i \leqslant l-1,$$

implying that for *any* w_j in R, we have, for $1 \leqslant i, j \leqslant l-2$,

$$\tau'_{i,j} = \sum_{1 \leqslant p \leqslant l} (a_{i,p} - (\tau_{i,1}|\tau_{l-1,1})a_{l-1,p})(v_{p,j} - w_j u_{p,1}).$$

Since $\|\tau_{l-1,1}\|_\infty \geqslant 1/(4T^{1/(l-1)})$ by (16), we can choose w_j in R to satisfy the condition

$$\left\| \sum_{1 \leqslant p \leqslant l} a_{l-1,p}(v_{p,j} - w_j u_{p,1}) \right\|_\infty = \left\| \sum_{1 \leqslant p \leqslant l} a_{l-1,p} v_{p,j} - w_j \tau_{l-1,1} \right\|_\infty < c_4/(4T^{1/(l-1)}) \tag{18}$$

where $c_4 = \max_{1 \leqslant i \leqslant d} \|\omega_i\|_\infty$. From this together with (16) and (17), we deduce that, for a constant $c_5 > 0$,

$$\|w_j\|_\infty \leqslant c_5 T^{v_{l-2} + 1/(l-1)} (1 \leqslant j \leqslant l-2).$$

Further, on setting

$$u_{p,j+1} := v_{p,j} - w_j u_{p,1}\ (1 \leqslant p \leqslant l;\ 1 \leqslant j \leqslant l-2),$$

we also obtain

$$\begin{aligned} \|u_{p,j+1}\|_\infty &\leqslant c_6 (T^{v_{l-2}} + T^{v_{l-2} + 1/(l-1) + (1+v/(l-1))}) \\ &\leqslant 2c_6 T^{v_{l-1}} \end{aligned} \tag{19}$$

for a constant c_6, with $v_{l-1} := v_{l-2} + 1 + (1+v)/(l-1)$. Setting $v_1 = 1 + v/(l-1)$, we have inductively $v_{l-1} = l + v - 1/(l-1)$. Since $\|\tau_{l-1,1}\|_\infty \geqslant \|\tau_{i,1}\|_\infty$ for $1 \leqslant i \leqslant l-1$, we have

$$\begin{aligned} \left\| \sum_{1 \leqslant p \leqslant l} a_{i,p} u_{p,j+1} \right\|_\infty &\leqslant \left\| \sum_{1 \leqslant p \leqslant l} (a_{i,p} - (\tau_{i,1}/\tau_{l-1,1})a_{l-1,p}) u_{p,j+1} \right\|_\infty \\ &\quad + \|\tau_{i,1}\|_\infty / \|\tau_{l-1,1}\|_\infty \left\| \sum_{1 \leqslant p \leqslant l} a_{l-1,p} u_{p,j+1} \right\|_\infty \\ &< c_4/(2T^{1/(l-1)}) \end{aligned}$$

in view of (17) and (18). Taking ${}^t\mathbf{u}_1$ and ${}^t\mathbf{u}_j = (u_{1,j} \ldots u_{l,j})$ from above for $2 \leqslant j \leqslant l-1$, the assertions of theorem 4 follow from the preceding inequality together with (16), (17) and (19), except for $(\tau_{j,p})$ being invertible which we verify in a moment. Indeed, for $1 \leqslant i, j \leqslant l-2$, we have

$$\tau_{i,j+1} = \sum_{1 \leqslant p \leqslant l} a_{i,p} u_{p,j+1} = \sum_{1 \leqslant p \leqslant l} a_{i,p}(v_{p,j} - w_j u_{p,1}) = \tau'_{i,j} + \rho_j \tau_{i,1}$$

where $\rho_j = (1/\tau_{l-1,1}) \sum_{1 \leqslant p \leqslant l} a_{l-1,p} v_{p,j} - w_j$ and further $\tau_{l-1,j+1} = \rho_j \tau_{l-1,1}$. The $l-1$ columns of $(\tau_{i,j})$ thus form the matrix with $(\tau_{1,1} \ldots \tau_{l-1,1})$ as its first row and $(\tau'_{1,j} + \rho_j \tau_{1,1} \ldots \tau'_{l-2,j} + \rho_j \tau_{l-2,1}\ \rho_j \tau_{l-1,1})$ for $1 \leqslant j \leqslant l-2$ as its next $l-2$ rows. The determinant of this matrix is, verified easily to be the same as $\pm \tau_{l-1,1} \times \det(\tau'_{i,j}) \neq 0$.

Remarks. (i) In view of corollary 7D of Schmidt [9], theorem 4 above certainly applies to the system $\vartheta_i x_1 - y_i$, $1 \leqslant i \leqslant l-1$ where $\vartheta_1, \ldots, \vartheta_{l-1}$ are real algebraic numbers such that $1, \vartheta_1, \ldots, \vartheta_{l-1}$ are linearly independent over $\boldsymbol{Q}$.

(ii) For general systems of $l-1$ real linear forms in l variables with rationality rank $l-1$, it does not seem to be possible, in general, to obtain independent integral vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{l-1}$ satisfying the required linear inequalities, along with a *polynomial* bound in T for $\|\mathbf{u}_1\|_\infty, \ldots, \|\mathbf{u}_{l-1}\|_\infty$, unless some assumption on the proper index as in theorem 4 were to be imposed. It is a question of obtaining explicit estimates for the quantity ω occurring in the proof of Kronecker's theorem (theorem 1 on page 458 of Lekkerkerker's interesting book [7]) in terms of the given ε. (For example, in the 2-dimensional case, Minkowski's theorem leads to $\omega \leqslant c/\varepsilon$ for a constant c. For higher dimensions, one can perhaps assert only the *existence* of a *finite* number $\omega = \omega(\varepsilon)$ depending on the given system of linear forms and the given ε. In this sense, one can get a version of theorem 4 with $\|\mathbf{u}_p\|_\infty \leqslant T^{1-1/(l-1)}\,(\omega(T^{-1/(l-1)}))^{l-1}$ for $1 \leqslant p \leqslant l-1$. For a system $\vartheta_i x_1 - x_{i+1}$, $1 \leqslant i \leqslant l-1$ as above and given ε, there can be seen to exist, in the notation of theorem 1, integral $a_1, \ldots, a_l$ not all 0, such that $|\vartheta_i a_1 - a_{i+1}|$ $\leqslant c_1\varepsilon (1 \leqslant i \leqslant l-1)$ and $|a_1| \leqslant c_2\varepsilon/h(\varepsilon^{-1})$ with $h(u) := \sup_{|a_1| \leqslant u} (|a_1|/\mathscr{T}(\mathbf{a}'))$. Using the result of Bombieri and Vaaler, one can thus obtain some adelic version of theorem 1 in [7] (p. 458).

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